

SOME EXISTENCE AND REGULARITY RESULTS FOR POROUS MEDIA AND FAST DIFFUSION EQUATIONS WITH A GRADIENT TERM

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ABSTRACT. In this paper we consider the problem

$$(P) \quad \begin{cases} u_t - \Delta u^m &= |\nabla u|^q + f(x, t), & u \geq 0 \text{ in } \Omega_T \equiv \Omega \times (0, T), \\ u(x, t) &= 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x), & x \in \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded regular domain, $1 < q \leq 2$, and $f \geq 0$, $u_0 \geq 0$ are in a suitable class of functions.

We obtain some results for elliptic-parabolic problems with measure data related to problem (P) that we use to study the existence of solutions to problem (P) according with the values of the parameters q and m .

To the memory of Juan Antonio Aguilar, our dearest friend.

1. INTRODUCTION

In this paper we will study the problem,

$$(1) \quad \begin{cases} u_t - \Delta u^m &= |\nabla u|^q + f(x, t), & u \geq 0 \text{ in } \Omega_T \equiv \Omega \times (0, T), \\ u(x, t) &= 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x), & x \in \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded regular domain, $m > 0$, $1 < q \leq 2$, and $f \geq 0$, $u_0 \geq 0$ are in a suitable class of functions. If $m > 1$, problem (1) is a model of growth in a porous medium, see for instance [5].

We refer to the fundamental monograph by J.L. Vázquez, [25], and the references therein for the basic results about *Porous Media Equations* (PME) and *Fast Diffusion Equation* (FDE) without gradient term. In [7] can be seen a optimal existence result for the Cauchy problem for the homogeneous (PME).

In this work we are interested in the existence and regularity of solutions to Problem (1) related to the parameters q and m . Some results were obtained in [17], [19], for $q = 2$ and bounded data. One of the main new features of this paper is to study general class of data according to the values of the parameters in the problems. This study is in some way motivated by the reference [2], where the stationary problem is analyzed.

A pertinent formal remark is the following:

The equation $u_t - \Delta u^m = \mu$, after the change $v = u^m$ becomes

$$b(v)_t - \Delta v = \mu \text{ with } b(s) = s^{\frac{1}{m}}.$$

The last formulation usually is known in the literature as *elliptic-parabolic equation*. References for problems related to these equations are [3], [8], [9] [11], [14], [15] and [24] among others.

We will study the elliptic-parabolic problems with μ a bounded Radon measure, which is the natural data in the application to the analysis of problem (1).

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The strategy that we follow in this work can be summarized in the following points.

- (1) We consider approximated problems that kill the degeneration or the singularity in the principal part ((PME) or (FDE) respectively) and truncating the first order term in the right hand side. With respect to these approximated problems, the existence of a solution follows using the well known results obtained in [12]. Here the natural setting is to find a *weak solution* in the sense of Definition 2.2 (where is formulated for the corresponding elliptic-parabolic equation).
- (2) We obtain uniform estimates of the solution of the approximated problems in such a way that the first order part in the second member is uniformly bounded in $L^1(\Omega_T)$.
- (3) The previous step motivates the study of a problem with measure data. To have more flexibility in the calculation we consider the corresponding formulation as elliptic-parabolic equation and looking for a *reachable solution* in the sense of Definition 2.4. One of the new features in this work, is the proof of the almost everywhere convergence of the gradient of the solutions of the approximated problems.
- (4) The final step is to use the uniform estimates and the a.e. convergence of the gradients to prove that, up to a subsequence, the second members of the approximated problems converge strongly in $L^1_{loc}(\Omega_T)$. That is, we find a *distributional solution*.

The organization of the paper is as follows. In Section 2, we prove the existence of reachable solutions for a class of *elliptic-parabolic* problem with measure data, including the corresponding cases to the (PME) and (FDE). To obtain the existence of reachable solutions, we show some a priori estimates for the solutions of the truncated problems and the pointwise convergence of the gradients, that allow us to conclude. In the last part of Section 2, as an application, we find existence of solution for the porous media and fast diffusion equations with a Radon measure data. A key point is the proof of a.e. convergence of the gradients, which will be used in the next section. It is worthy to point out that these results improve in some way the ones obtained in [23] for $m > 1$, and give a proof for the (FDE) with measure data.

Section 3 deals with the (PME) with a gradient term. We use the strategy describe above. In the first part of the section we get the existence results for the interval $1 < m \leq 2$. The main result in this case is Theorem 3.1. In the last part of this section we consider the complementary interval of m , that is, $m > 2$. In this case we are able to prove the existence of a solution with $L^1(\Omega_T)$ data. This result is the content of Theorem 3.3.

In the last part of the Section 3, we point out the particular behavior of the case $p = 2$, $m = 2$. Indeed, if the source term $f = 0$, via a change of variable, we show that it is equivalent to a (PME) equation with $m = \frac{5}{3}$. As a consequence, we obtain in this case the *finite speed of propagation* property, and also a selfsimilar solution with compact support in each fixed positive time.

Finally, in Section 4 we analyze the fast diffusion equation, i.e. $0 < m < 1$, with a convenient hypothesis of integrability of the source term. The main result of this section is Theorem 4.1. We also prove Theorem 4.3 that gives us the *finite extinction time* property of regular solution (see Definition 4.2) if $0 < m < 1$ and $q = 2$.

We have tried to write the paper in an almost self-contained form, moreover we give precise references for all the points that are not detailed in the work.

In a forthcoming paper we will obtain some results of non-uniqueness for quadratic growth therm.

2. SOME RESULTS FOR AN *elliptic-parabolic* PROBLEM WITH MEASURE DATA

We will consider the general problem

$$(2) \quad \begin{cases} (b(v))_t - \Delta v &= \mu & \text{in } \Omega \times (0, T), \\ v(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ b(v(x, 0)) &= b(v_0(x)) & \text{in } \Omega, \end{cases}$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is continuous strictly increasing function such that $b(0) = 0$, $b(v_0) \in L^1(\Omega)$, $f \in L^\infty(\Omega_T)$ and μ is a Radon measure whose total variation is finite in Ω_T . We will assume the following hypotheses on b :

$$(B) \quad \begin{cases} (B1) \text{ There exists } a_1 > 0 \text{ such that } b(s) \geq Cs^{a_1} \text{ for } s \gg 1, \\ (B2) \text{ There exists } a_2 < 1 \text{ such that } |b'(s)| \leq \frac{1}{s^{a_2}} \text{ for } s \ll 1, \\ (B3) \text{ There exists } a_3 \in (\frac{N-1}{N}, 1) \text{ such that} \\ \quad \text{Eithr } b' \in \mathcal{C}([0, \infty)) \text{ and } |b'(s)|b^{2a_3-1}(s) \leq s^{\frac{N+2a_1}{N}-\varepsilon} \text{ as } s \rightarrow \infty \text{ or } |b'(s)| \leq b^{2-2a_3-\varepsilon}(s) \text{ as } s \rightarrow \infty. \end{cases}$$

Remark 2.1. *The following examples of b will be considered in this work.*

$$(1) \quad b(s) = (s + \frac{1}{n})^\sigma - (\frac{1}{n})^\sigma \text{ if } s \geq 0, \text{ for some } \sigma > \frac{(N-2)_+}{N} \text{ and } 1 \leq n < \infty ;$$

$$(2) \quad b(s) = \frac{1}{m} \int_0^s (H^{-1}(\sigma))^{\frac{1}{m}-1} d\sigma = \frac{1}{m} \int_0^{H^{-1}(s)} \sigma^{\frac{1}{m}-1} H'(\sigma) d\sigma, \text{ where}$$

$$H(s) = \frac{4}{5} s^{\frac{5}{4}} \text{ if } m = 2, \quad H(s) = \int_0^s e^{\frac{t-\frac{2-m}{m}}{m(2-m)}} dt \text{ if } 0 < m < 2.$$

For $\mu \in L^\infty(\Omega_T)$ and $b(v_0) \in L^\infty(\Omega)$, we mean a solution in the sense of the following definition.

Definition 2.2. *Assume that $\mu \in L^\infty((0, T); L^\infty(\Omega))$ and $b(v_0) \in L^\infty(\Omega)$. We say that v is a weak solution to (2) if*

- (1) $v \in L^2((0, T); W_0^{1,2}(\Omega)) \cap L^\infty((0, T); L^\infty(\Omega))$,
- (2) The function $b(v) \in \mathcal{C}((0, T); L^q(\Omega))$ for all $q < \infty$,
- (3) $(b(v))_t \in L^2((0, T); W^{-1,2}(\Omega))$.

And for every $\phi \in L^2((0, T); W_0^{1,2}(\Omega))$ the following identity holds,

$$(3) \quad \int_0^T \langle b(v)_t, \phi \rangle + \iint_{\Omega_T} \nabla v \cdot \nabla \phi = \iint_{\Omega_T} \mu \phi.$$

The following result is well known .

Theorem 2.3. *Assuming $\mu \in L^\infty((0, T); L^\infty(\Omega))$ and $b(v_0) \in L^\infty(\Omega)$, there exists a unique weak solution to problem (2) in the sense of Definition 2.2.*

The proof of Theorem 2.3 follows closely the argument developed in [3] and [15].

2.1. Reachable solutions. We now introduce the notion of solution for problem (2) natural to consider measure data. For elliptic equations this notion was introduced in [18]. We refer to [16] for the parabolic equation. See also [1] for some particular cases.

In the case of $\mu \in L^1(\Omega_T)$, the renormalized solution is studied in [11]. We give a complete analysis under the (B) conditions in order to obtain some regularity on spatial gradient of the solutions.

Definition 2.4. Assume that μ is a Radon measure whose total variation is finite in Ω_T and $b(v_0) \in L^1(\Omega)$.

We say that v is a reachable solution to (2) if

- (1) $T_k(v) \in L^2((0, T); W_0^{1,2}(\Omega))$ for all $k > 0$.
- (2) For all $t > 0$ there exist both one-side limits $\lim_{\tau \rightarrow t^\pm} b(v(\cdot, \tau))$ weakly- $*$ in the sense of measures.
- (3) $b(v(\cdot, t)) \rightarrow b(v_0(\cdot))$ weakly- $*$ in the sense of measures as $t \rightarrow 0$.
- (4) There exist three sequences $\{v_n\}_n$ in $L^2((0, T); W_0^{1,2}(\Omega))$, $\{h_n\}_n$ in $L^\infty((0, T); L^\infty(\Omega))$ and $\{g_n\}_n$ in $L^\infty(\Omega)$ such that if v_n is the weak solution to problem

$$(4) \quad \begin{cases} (b(v_n))_t - \Delta v_n &= h_n & \text{in } \Omega \times (0, T), \\ v_n(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ v_n(x, 0) &= b^{-1}(g_n(x)) & \text{in } \Omega, \end{cases}$$

then

- (a) $g_n \rightarrow b(v_0)$ in $L^1(\Omega)$.
- (b) $h_n \xrightarrow{*} \mu$ as measures.
- (c) $\nabla v_n \rightarrow \nabla v$ strongly in $L^\sigma((0, T); L^\sigma(\Omega))$ for $1 \leq \sigma < \frac{N + 2a_1}{N + a_1}$.
- (d) The sequence $\{b(v_n)\}_n$ is bounded in $L^\infty((0, T); L^1(\Omega))$ and $b(v_n) \rightarrow b(v)$ strongly in $L^1(\Omega_T)$.

To prove the existence of a reachable solution, we need the following Lemma whose proof can be obtained by approximation.

Lemma 2.5. Let $v \in L^2(0, T; W_0^{1,2}(\Omega))$ satisfy $b(v)_t \in L^{2'}(0, T; W^{-1,2'}(\Omega))$. Assume that $\phi(s) : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz-continuous function such that $\phi(0) = 0$. Then, if we define

$$(5) \quad \Phi(s) = \int_0^s b'(\sigma) \phi(\sigma) d\sigma,$$

the following integration by parts formula holds:

$$(6) \quad \int_{t_1}^{t_2} \langle b(u)_t, \phi(u) \rangle_{W^{-1,2'}(\Omega), W_0^{1,2}(\Omega)} dt = \int_\Omega \Phi(u(x, t_2)) dx - \int_\Omega \Phi(u(x, t_1)) dx,$$

for every $0 \leq t_1 < t_2$.

We will now prove the existence of a reachable solution to Problem (4).

2.2. Some a priori estimates. Let us consider the following approximating problems:

$$(7) \quad \begin{cases} b(v_n)_t - \Delta v_n &= h_n, & (x, t) \in \Omega_T, \\ v_n(x, t) &= 0, & (x, t) \in \partial\Omega \times (0, T), \\ v_n(x, 0) &= b^{-1}(g_n(x)), & x \in \Omega, \end{cases}$$

where $g_n \rightarrow b(v_0)$ strongly in $L^1(\Omega)$ and $h_n \rightarrow \mu$ in the weak-* sense in Ω_T . The existence of weak solutions to these problems follows from Theorem 2.3.

Let us begin by proving the next proposition.

Proposition 2.6. *Let $\{v_n\}_n$ be a sequence of solutions of the approximate problems (7). Then*

- (1) *For each $0 < \beta < \frac{1}{2}$, the sequence $\{(|v_n| + 1)^\beta - 1\}_n$ is bounded in $L^2(0, T; W_0^{1,2}(\Omega))$*
- (2) *The sequence $\{b(v_n)\}_n$ is bounded in the space $L^\infty(0, T; L^1(\Omega))$ and $\{(b(v_n))_t\}_n$ is bounded in $L^1(\Omega_T) + L^\sigma(0, T; W^{-1,\sigma}(\Omega))$, for some $\sigma > 1$.*

Moreover,

$$(8) \quad \iint_{\Omega_T} \frac{|\nabla v_n|^2}{(1 + |v_n|)^{\alpha+1}} \leq C \quad \text{for all } \alpha > 0.$$

Furthermore, the sequence $\{|\nabla v_n|\}_n$ is bounded in the Marcinkiewicz spaces $M^q(\Omega_T)$ where $q = \frac{(N+2a_1)}{N+a_1}$ and $\{v_n\}_n$ is bounded in the space $M^\sigma(\Omega_T)$ where $\sigma = \frac{N+2a_1}{N}$.

PROOF. Take $\phi(v_n) \equiv T_k(v_n)$ in Lemma 2.5, with $k > 0$, as test function in the weak formulation of (7), then

$$(9) \quad \int_{\Omega} \Phi_k(v_n(x, t)) dx - \int_{\Omega} \Phi_k(b^{-1}(g_n(x))) dx + \iint_{\Omega_T} |\nabla T_k(v_n)|^2 \leq k|\mu|(\Omega_T),$$

where $\Phi_k(s) = \int_0^s T_k(\sigma) b'(\sigma) d\sigma$ which is a nonnegative function. Since $\Phi_k(s) \leq k|b(s)|$, it follows from (9) that

$$(10) \quad \int_{\Omega} |\Phi_k(v_n(x, t))| dx + \iint_{\Omega_T} |\nabla T_k(v_n)|^2 \leq ck \iint_{\Omega_T} (1 + |b(v_n)|) + ck.$$

Now, dropping a nonnegative term, dividing by k and letting k go to 0, it yields

$$\int_{\Omega} |b(v_n(x, t))| dx \leq c \iint_{\Omega_T} |b(v_n)| + c.$$

Thus, Gronwall's Lemma implies that

$$(11) \quad \sup_{t \in [0, T]} \int_{\Omega} |b(v_n(x, t))| dx \leq C.$$

Moreover, going back to (10) we get

$$(12) \quad \iint_{\Omega_T} |\nabla T_k(v_n)|^2 \leq Ck.$$

In order to prove the next estimate, we define $\theta(s) = (1 - (1 + |s|)^{-\alpha})$, with $0 < \alpha$, and we take $\theta(v_n)$ as test function in the approximating problems. There results,

$$(13) \quad \int_{\Omega} \Theta(v_n(T)) dx - \int_{\Omega} \Theta(b^{-1}(g_n)) dx + \alpha \iint_{\Omega_T} \frac{|\nabla v_n|^2}{(1 + |v_n|)^{\alpha+1}} \leq \|\mu\|,$$

where $\Theta(s) = \int_0^s b'(\sigma) \theta(\sigma) d\sigma$. Hence

$$\iint_{\Omega_T} \frac{|\nabla v_n|^2}{(1 + |v_n|)^{\alpha+1}} \leq C \text{ for all } \alpha > 0$$

and then $\{(|v_n| + 1)^\beta - 1\}_n$ is bounded in $L^2(0, T; W_0^{1,2}(\Omega))$ for all $0 < \beta < \frac{1}{2}$.

To get the estimates on the Marcinkiewicz spaces, we follow closely the arguments in [6], see also [4].

From hypothesis (B1), we obtain that $b(s) \geq C_1 s^{a_1} - C$ for all $s \geq 0$. By using again (11) we get the existence of a positive constant C such that,

$$(14) \quad |\{x \in \Omega : |v_n(x, t)| > k\}| \leq \frac{C}{k^{a_1}} \text{ for almost all } t \in [0, T], \text{ all } k > 0 \text{ and all } n \in \mathbb{N}.$$

Thus, by Sobolev's inequality and (12),

$$(15) \quad \int_0^T (|\{x \in \Omega : |v_n(x, t)| \geq k\}|)^{2/2^*} dt \leq \int_0^T \left(\frac{\|T_k(v_n(x, t))\|_{2^*}^2}{k^{2^*}} \right)^{2/2^*} dt \\ \leq C \int_0^T \frac{\|\nabla T_k(v_n(x, t))\|_2^2}{k^2} dt \leq \frac{C}{k} \text{ for all } k > 0 \text{ and all } n \in \mathbb{N}.$$

Therefore, combining (15) and (14) we obtain, for all $k > 0$ and all $n \in \mathbb{N}$,

$$(16) \quad |\{(x, t) \in \Omega_T : |v_n(x, t)| \geq k\}| \\ = \int_0^T (|\{x \in \Omega : |v_n(x, t)| \geq k\}|)^{1-(2/2^*)} (|\{x \in \Omega : |v_n(x, t)| \geq k\}|)^{2/2^*} dt \leq \frac{C}{k^{\frac{N+2a_1}{N}}}.$$

A similar estimate for the gradients is now easy to obtain. Indeed, for every $h, k > 0$, we have

$$|\{(x, t) \in \Omega_T : |\nabla v_n(x, t)| \geq h\}| \\ \leq |\{(x, t) \in \Omega_T : |v_n(x, t)| \geq k\}| + |\{(x, t) \in \Omega_T : |\nabla T_k(v_n(x, t))| \geq h\}| \\ \leq \frac{C}{k^{\frac{N+2a_1}{N}}} + \iint_{\Omega_T} \frac{|\nabla T_k(v_n)|^2}{h^2} \leq \frac{C}{k^{\frac{N+2a_1}{N}}} + \frac{Ck}{h^2}.$$

Minimizing in k , we obtain that for $k = h^{\frac{N}{N+a_1}}$,

$$(17) \quad |\{(x, t) \in \Omega_T : |\nabla v_n(x, t)| \geq h\}| \leq \frac{C}{h^{\frac{N+2a_1}{N+a_1}}} \text{ for all } h > 0 \text{ and all } n \in \mathbb{N}.$$

Hence,

$$(18) \quad \iint_{\Omega_T} |v_n|^\rho \leq C \text{ for all } 0 < \rho < \frac{N+2a_1}{N}$$

$$(19) \quad \iint_{\Omega_T} |\nabla v_n|^r \leq C \text{ for all } 0 < r < \frac{N+2a_1}{N+a_1}.$$

From equations (7) we obtain, $\{(b(v_n))_t\}_n$ is bounded in $L^1(\Omega_T) + L^\sigma(0, T; W^{-1, \sigma}(\Omega))$ for some $\sigma \geq \frac{N+2a_1}{a_1}$. Consider ϱ such that

$$(20) \quad -\Delta \varrho = 1 \text{ in } \Omega, \varrho \in W_0^{1,2}(\Omega).$$

We claim that for all $0 < \delta < \min\{1, \frac{1}{a_2}\}$,

$$(21) \quad \int_0^T \int_\Omega \frac{|\nabla v_n|^2}{(v_n + \frac{1}{n})^{1+\delta}} \varrho \leq C \text{ uniformly in } n.$$

To prove the claim we define $K(s) = \int_0^s \frac{b'(\sigma)}{(\sigma + \frac{1}{n})^\delta} d\sigma$. Using (B2) we get easily that $K(v_n)$ is well defined, $K(0) = 0$ and

$$(22) \quad \sup_{t \in [0, T]} \int_\Omega K(v_n(x, t)) \varrho dx \leq C.,$$

Using $\frac{\varrho}{(v_n + \frac{1}{n})^\delta}$ as a test function in (7), it follows that

$$\begin{aligned} \int_\Omega K(v_n(x, T)) \varrho dx + \frac{1}{1-\delta} \iint_{\Omega_T} \left[(v_n + \frac{1}{n})^{1-\delta} - (\frac{1}{n})^{1-\delta} \right] (-\Delta \varrho) = \\ \iint_{\Omega_T} \frac{|\nabla v_n|^2}{(v_n + \frac{1}{n})^{1+\delta}} \varrho + \iint_{\Omega_T} \frac{h_n}{(v_n + \frac{1}{n})^{1+\delta}} \varrho + \int_\Omega K(v_n(x, 0)) \varrho dx. \end{aligned}$$

Since $0 < \delta < 1$, then using (18), we get

$$\frac{1}{1-\delta} \iint_{\Omega_T} \left[(v_n + \frac{1}{n})^{1-\delta} - (\frac{1}{n})^{1-\delta} \right] \leq C \text{ uniformly in } n.$$

As a consequence, and by (22),

$$(23) \quad \iint_{\Omega_T} \frac{|\nabla v_n|^2}{(v_n + \frac{1}{n})^{1+\delta}} \varrho \leq C_1 \text{ uniformly in } n.$$

In particular the claim follows.

In the same way, using $1 - \frac{1}{(b(v_n) + 1)^\delta}$, where $0 < \delta$, as a test function in (7), it follows that

$$(24) \quad \sup_{t \in [0, T]} \int_\Omega b(v_n(x, t)) dx + \int_0^{T_1} \int_\Omega \frac{b'(v_n) |\nabla v_n|^2}{(b(v_n) + 1)^{1+\delta}} \leq C \text{ uniformly in } n.$$

To finish we have just to prove that

$$(25) \quad b(v_n) \rightarrow b(v) \text{ strongly in } L^1(\Omega_T).$$

Notice that $\|b(v_n)\|_{L^1(\Omega_T)} \leq C$ and $b(v_n) \rightarrow b(v)$ e.a in Ω_T .

If $b(s) \leq C s^{\frac{N+2a_1}{N}-\varepsilon}$, for some $\varepsilon > 0$, as $s \rightarrow \infty$, then using (18) and by Vitali's lemma we reach the strong convergence in (25).

Assume that the condition B_3 holds, let $w_n = (b(v_n) + 1)^\beta$, where $\beta \leq 1$ to be chosen later, then $\|w_n\|_{L^\infty(0,T;L^1(\Omega))} \leq C$ and $\nabla w_n = \beta b'(v_n)(b(v_n) + 1)^{\beta-1} \nabla v_n$. Thus, for $\delta > 0$, we have

$$\begin{aligned} \iint_{\Omega_T} |\nabla w_n| &= \beta \iint_{\Omega_T} b'(v_n)(b(v_n) + 1)^{\beta-1} |\nabla v_n| = \beta \iint_{\Omega_T} b'(v_n)(b(v_n) + 1)^{\frac{2\beta+\delta-1}{2}} \frac{|\nabla v_n|}{(b(v_n) + 1)^{\frac{\delta+1}{2}}} \\ &\leq \frac{1}{2} \beta \iint_{\Omega_T} \frac{b'(v_n) |\nabla v_n|^2}{(b(v_n) + 1)^{\delta+1}} + \frac{1}{2} \beta \iint_{\Omega_T} b'(v_n)(b(v_n) + 1)^{2\beta+\delta-1}. \end{aligned}$$

From (24), we obtain that

$$\iint_{\Omega_T} \frac{b'(v_n) |\nabla v_n|^2}{(b(v_n) + 1)^{\delta+1}} \leq C.$$

Now, if the first condition in B_3 holds, then choosing δ small enough and $\beta = a_3$, we reach that

$$b'(s)(b(s) + 1)^{2\beta+\delta-1} \leq C_1 s^{\frac{N+2a_1}{N}-\varepsilon} \text{ as } s \rightarrow \infty \text{ for some } \varepsilon > 0.$$

Thus

$$\frac{\beta}{2} \iint_{\Omega_T} b'(v_n)(b(v_n) + 1)^{2\beta+\delta-1} \leq C \text{ uniformly in } n.$$

If the second condition in B_3 holds, with the same choice of β and for δ small,

$$b'(s)(b(s) + 1)^{2\beta+\delta-1} \leq b^{1-\varepsilon}(s) \text{ as } s \rightarrow \infty,$$

and hence

$$\frac{\beta}{2} \iint_{\Omega_T} b'(v_n)(b(v_n) + 1)^{2\beta+\delta-1} \leq C \text{ uniformly in } n.$$

Therefore, we conclude that $\|w_n\|_{L^1(0,T;W^{1,1}_0(\Omega))} \leq C$ for all n . Hence, using the Gagliardo-Nirenberg inequality we conclude that $\|w_n\|_{L^{\frac{N}{N-1}}(\Omega_T)} \leq C$.

Since $\beta \frac{N}{N-1} > 1$, then $\{b(v_n)\}_n$ is bounded in $L^{1+\varepsilon}(\Omega_T)$ for some $\varepsilon > 0$. Hence, using Vitali's lemma, we obtain that the sequence $\{b(v_n)\}_n$ is compact in $L^1(\Omega_T)$ and so we may extract a subsequence (also labelled by n) such that $b(v_n) \rightarrow b(v)$ strongly in $L^1(\Omega_T)$. ■

Remark 2.7. If $b(s) = (s + \frac{1}{n})^\sigma - (\frac{1}{n})^\sigma$, then

- (1) If $\sigma \leq \frac{N}{N-1}$, we can prove that the sequence $\{(b(v_n))\}_n$ is bounded in $L^1(0,T;W^{1,1}_{loc}(\Omega))$. This follows using estimates (24).
- (2) If $\sigma \leq \frac{N}{N-2}$, we can prove that the sequence $\{(b(v_n))\}_n$ converge strongly in $L^1(\Omega_T)$. This follows using (18).

2.3. Pointwise convergence of the gradients. In this subsection we prove that up to a subsequence

$$\nabla v_n \rightarrow \nabla v, \text{ a.e. in } \Omega_T \text{ as } n \rightarrow \infty, .$$

Hence we will obtain that the sequence $\{v_n\}_n$ satisfies condition (4) (c) in Definition 2.4.

Proposition 2.8. Consider $\{v_n\}_n$, the solution of the approximated problems (7). Then, up to subsequence,

$$(26) \quad \nabla T_k(v_n) \rightarrow \nabla T_k(v) \quad \text{almost everywhere in } \Omega_T.$$

As a consequence, $\nabla v_n \rightarrow \nabla v$ almost everywhere in Ω_T .

PROOF. We recall the time-regularization of functions due to Landes and Mustonen (see [21], [22]). Consider w such that $T_k(w) \in L^2(0, T; W_0^{1,2}(\Omega)) \cap \mathcal{C}([0, T]; L^2(\Omega))$. For every $\nu \in \mathbb{N}$, we define $(T_k w)_\nu$ as the solution of the Cauchy problem

$$(27) \quad \begin{cases} \frac{1}{\nu}[(T_k w)_\nu]_t + (T_k w)_\nu = T_k w; \\ (T_k w)_\nu(0) = T_k(w_0). \end{cases}$$

Then, one has, $(T_k w)_\nu \in L^2(0, T; W_0^{1,2}(\Omega))$, $((T_k w)_\nu)_t \in L^2(0, T; W_0^{-1,2}(\Omega))$,

$$\|(T_k w)_\nu\|_{L^\infty(\Omega_T)} \leq \|T_k w\|_{L^\infty(\Omega_T)} \leq k,$$

and as ν goes to infinity, $(T_k w)_\nu \rightarrow T_k w$ strongly in $L^2(0, T; W_0^{1,2}(\Omega))$.

From now on, we will denote by $\omega(n, \nu, \varepsilon)$ any quantity such that

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{\nu \rightarrow \infty} \limsup_{n \rightarrow \infty} |\omega(n, \nu, \varepsilon)| = 0.$$

Taking $T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu)$ as test function in (7), we obtain

$$\begin{aligned} & \int_0^T \langle (b(v_n))_t, T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \rangle dt + \iint_{\Omega_T} \nabla v_n \cdot \nabla T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \\ &= \iint_{\Omega_T} h_n T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \end{aligned}$$

We will analyze the integrals which appear in the previous equality. For simplicity of typing we set $w_n = b(v_n)$, $w = b(v)$ and $\psi = b^{-1}$.

It is clear that

$$\iint_{\Omega_T} h_n T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \leq C\varepsilon.$$

Notice that

$$\int_0^T \langle (b(v_n))_t, T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \rangle dt = \int_0^T \langle (w_n)_t, T_\varepsilon(w_n - (T_k(w))_\nu) \rangle dt.$$

Then using the same arguments as in [13] (see also (3.37) in [1]) we reach that

$$(28) \quad \int_0^T \langle (b(v_n))_t, T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu) \rangle dt \geq \omega(n, \nu, \varepsilon).$$

We set

$$I = \iint_{\Omega_T} \nabla v_n \cdot \nabla T_\varepsilon(b(v_n) - (T_k(b(v)))_\nu).$$

Claim. It holds that

$$(29) \quad I \geq \iint_{\{|w_n| \leq k\}} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla T_\varepsilon(T_k(w_n) - (T_k w)_\nu) + \omega^\varepsilon(n, \nu).$$

Indeed,

$$\begin{aligned}
I &= \iint_{\{|w_n| \leq k\}} \psi'(w_n) \nabla T_k(w_n) \cdot \nabla T_\varepsilon(T_k(w_n) - (T_k w)_\nu) \\
&+ \iint_{\{|w_n| > k\}} \psi'(w_n) \nabla w_n \cdot \nabla T_\varepsilon(w_n - (T_k w)_\nu) \\
&\geq \iint_{\{|w_n| \leq k\}} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla T_\varepsilon(T_k(w_n) - (T_k w)_\nu) \\
&- \iint_{\{|w_n| > k, |w_n - (T_k w)_\nu| \leq \varepsilon\}} \psi'(w_n) \nabla w_n \cdot \nabla (T_k w)_\nu.
\end{aligned}$$

Since $\|(T_k w)_\nu\|_\infty \leq k$, the last integrand is different from zero only in the set where $|w_n| \leq k + \varepsilon$, therefore the last integrand is bounded by

$$\begin{aligned}
&c_1(k) \left[\iint_{\Omega_T} |\nabla T_{k+\varepsilon} w_n|^2 \right]^{\frac{1}{2}} \left[\iint_{\Omega_T} |\nabla (T_k w)_\nu|^2 \chi_{\{|w_n| > k\}} \right]^{\frac{1}{2}} \\
&\leq c_2(k, \varepsilon) \left[\iint_{\Omega_T} |\nabla T_k w|^2 \chi_{\{|w| > k\}} \right]^{\frac{1}{2}} + \omega^\varepsilon(n, \nu) = \omega^\varepsilon(n, \nu).
\end{aligned}$$

Here we have used the a.e. convergence of $\chi_{\{|w_n| > k\}}$ to $\chi_{\{|w| > k\}}$ as $n \rightarrow +\infty$, which holds for all k (see for instance [13]).

Therefore, putting together the above estimates, we reach that

$$\begin{aligned}
(30) \quad &\iint_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla T_\varepsilon(T_k w_n - (T_k w)_\nu) \\
&= \iint_{\{|w_n| \leq k\}} \psi'(w_n) \nabla T_k(w_n) \cdot \nabla T_\varepsilon(w_n - (T_k w)_\nu) \leq \omega(n, \nu, \varepsilon).
\end{aligned}$$

Thus

$$(31) \quad \iint_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w_n - (T_k w)_\nu) \chi_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} \leq \omega(n, \nu, \varepsilon).$$

Hence it follows that

$$\begin{aligned}
&\iint_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w_n - T_k w) \chi_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} \\
&= \iint_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w_n - (T_k w)_\nu) \chi_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} \\
&+ \iint_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla ((T_k w)_\nu - T_k w) \chi_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} \\
&= \iint_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w_n - (T_k w)_\nu) \chi_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} + \omega(n, \nu, \varepsilon).
\end{aligned}$$

Therefore, by (31) we obtain

$$(32) \quad \iint_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k(w_n) \cdot \nabla (T_k w_n - T_k w) \chi_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} \leq \omega(n, \nu, \varepsilon).$$

On the other hand, we have that

$$\iint_{\Omega_T} \psi'(T_k(w_n)) \nabla T_k w \cdot \nabla (T_k w_n - T_k w) \chi_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} = \omega^{\nu, \varepsilon}(n).$$

Indeed, $\psi'(T_k(w_n)) \nabla T_k w \rightarrow \psi'(T_k(w)) \nabla T_k w$ strongly in $L^2(\Omega_T)$ and $\nabla T_k w_n \rightharpoonup \nabla T_k w$ weakly in $L^2(\Omega_T)$. Hence, we deduce from (32) that

$$(33) \quad \iint_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} \psi'(T_k(w_n)) |\nabla T_k(w_n) - \nabla T_k w|^2 \leq \omega(n, \nu, \varepsilon).$$

Denoting $\Psi_{n,k} = \psi'(T_k(w_n)) |\nabla T_k(w_n) - \nabla T_k w|^2$, then

$$(34) \quad \iint_{\{|T_k(w_n) - (T_k w)_\nu| \leq \varepsilon\}} \Psi_{n,k} \leq \omega(n, \nu, \varepsilon).$$

Since

$$(35) \quad \chi_{\{|T_k(w_n) - (T_k w)_\nu| > \varepsilon\}} \rightarrow \chi_{\{|T_k(w) - (T_k w)_\nu| > \varepsilon\}} \quad \text{strongly in } L^\rho(\Omega_T), \forall \rho \geq 1.$$

then using Hölder inequality and the fact that $\Psi_{n,k}$ is bounded in $L^1(\Omega_T)$, we deduce that

$$(36) \quad \iint_{\Omega_T} \Psi_{n,k}^\theta = \omega(n, \nu, \varepsilon) \text{ for all } \theta < 1. \text{ Since } \psi'(T_k(s)) \geq c(k)(\psi'(T_k(s)))^2, \text{ it follows that}$$

$$\lim_{n \rightarrow \infty} \iint_{\Omega_T} \left[(\psi'(T_k(w_n)))^2 |\nabla T_k(w_n) - \nabla T_k w|^2 \right]^\theta = 0,$$

Hence we get $\psi'(T_k(w_n)) \nabla T_k(w_n) \rightarrow \psi'(T_k(w)) \nabla T_k(w)$ e.a in Ω and then

$$\nabla \psi(T_k(w_n)) \rightarrow \nabla \psi(T_k(w)) \quad \text{a.e. in } \Omega.$$

Using the fact that ψ is a strictly monotone function we reach

$$\nabla T_k(w_n) \rightarrow \nabla T_k(w) \quad \text{a.e. in } \Omega$$

and then $\nabla T_k(w_n) \rightarrow \nabla T_k(w)$ a.e in Ω . ■

Corollary 2.9.

(1) Let $\{v_n\}_n$ be a sequence of solutions of (7), then $\nabla v_n \rightarrow \nabla v$ strongly in $L^\sigma(\Omega_T)$ for all $1 \leq \sigma < \frac{N+2a_1}{N+a_1}$.

(2) From estimate (36), we reach

$$(37) \quad \nabla T_k(v_n) \rightarrow \nabla T_k(v) \text{ strongly in } L^\sigma(\Omega_T) \text{ for all } \sigma < 2.$$

Corollary 2.10. The following equality holds

$$(38) \quad - \int_{\Omega} b(v_0(x)) \Phi(x, 0) dx - \iint_{\Omega_T} b(v) \Phi_t + \iint_{\Omega_T} \nabla v \cdot \nabla \Phi = \iint_{\Omega_T} \Phi d\mu,$$

for every $\Phi \in C^\infty(\overline{\Omega_T})$, with $\Phi(\cdot, t) \in C_0(\Omega)$ for all $t \in (0, T)$ and $\Phi(x, T) = 0$ for all $x \in \Omega$.

2.4. Application to the porous media and fast diffusion equations with a Radon measure.

The results in the above subsection allow us to consider the problem

$$(39) \quad \begin{cases} u_t - \Delta u^m &= \mu & \text{in } \Omega \times (0, T), \\ u(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

with $m > \frac{(N-2)_+}{N}$, $u_0 \in L^1(\Omega)$ and μ is a Radon measure whose total variation is finite in Ω_T .

In the case of the porous media equation, i.e., $m > 1$, the existence results are obtained in [23] by using some result in [20]. Here we extend the results to the interval $\frac{(N-2)_+}{N} < m < 10$, by proving, moreover, the a.e convergence of the gradients of the truncated problems to the gradient of the solution of problem (39). Our approach is alternative by using the *elliptic-parabolic* framework.

We will consider the approximated form

$$(40) \quad \begin{cases} u_{nt} - \operatorname{div}(m(u_n + \frac{1}{n})^{m-1} \nabla u_n) &= h_n & \text{in } \Omega \times (0, T), \\ u_n(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) &= T_n(u_0) & \text{in } \Omega. \end{cases}$$

The main goal of this subsection is to show compactness results for the sequences $\{|\nabla u_n|\}_n$ and $\{T_k(u_n)\}_n$ including the case $\frac{(N-2)_+}{N} < m < 1$.

Define $v_n \equiv (u_n + \frac{1}{n})^m - (\frac{1}{n})^m$, then v_n solves

$$(41) \quad \begin{cases} (b(v_n))_t - \Delta v_n &= h_n, & (x, t) \in \Omega_T, \\ v_n(x, t) &= 0, & (x, t) \in \partial\Omega \times (0, T), \\ v_n(x, 0) &= \varphi^{-1}(T_n(u_0(x))), & x \in \Omega, \end{cases}$$

where $b(s) = (s + (\frac{1}{n})^m)^{\frac{1}{m}} - \frac{1}{n}$.

Theorem 2.11. *Consider v_n , the solution to (41) and $u_n = b(v_n)$ that solves (40). There exists a measurable function u such that $u^m \in L^r(0, T; W_0^{1,r}(\Omega))$ for all $r < 1 + \frac{1}{Nm+1}$, and, up to a subsequence,*

- (1) $\nabla u_n \rightarrow \nabla u$ e.a in Ω_T and then $\nabla v_n \rightarrow \nabla v$ e.a in Ω_T where $v = u^m$.
- (2) $T_k(v_n) \rightarrow T_k(v)$ strongly in $L^\sigma(0, T; W_0^{1,\sigma}(\Omega))$ for all $k > 0$ and for all $\sigma < 2$.

PROOF. Since b verifies the condition (B) with $a_1 = \frac{1}{m}$, then by the results of Corollary 2.9, we obtain that

- (1) $\nabla v_n \rightarrow \nabla v$ strongly in the space $L^\sigma((0, T); L^\sigma(\Omega))$ for all $1 \leq \sigma < 1 + \frac{1}{Nm+1}$ and
- (2) $\nabla T_k(v_n) \rightarrow \nabla T_k(v)$ strongly in $L^\sigma(\Omega_T)$ for all $\sigma < 2$.

Let $\psi \in \mathcal{C}_0^\infty(\Omega_T)$ be such that $\psi \geq 0$ in Ω_T , we claim that

$$(42) \quad \left\{ \frac{|\nabla T_k(u_n)|}{(u_n + \frac{1}{n})^\theta} \psi \right\}_n \text{ is uniformly bounded in } L^1(\Omega_T) \text{ for some } \theta > 0.$$

To prove the claim we will consider separately the cases: $0 < m < 1$ and $m > 1$.

Let begin by the case $0 < m < 1$. Using $T_k(u_n)$ as a test function in (39), we get

$$\iint_{\Omega_T} (u_n + \frac{1}{n})^{m-1} |\nabla T_k(u_n)|^2 \leq C.$$

Thus the claim follows with $\theta = 1 - m > 0$.

We deal now with the case $m > 1$. Using $\frac{\psi}{(v_n + (\frac{1}{n})^m)^\delta}$, where $\delta < \frac{1}{m}$, as a test function in (41)

we obtain that

$$(43) \quad \iint_{\Omega_T} \frac{|\nabla v_n|^2}{(v_n + (\frac{1}{n})^m)^{\delta+1}} \psi \leq C.$$

Notice that

$$\frac{|\nabla T_k(u_n)|}{(u_n + \frac{1}{n})^\theta} \psi = \frac{|\nabla T_{k'} v_n|}{(v_n + (\frac{1}{n})^m)^{1-\frac{1-\theta}{m}}} \psi.$$

Now using (43) we conclude that $\left\{ \frac{|\nabla T_k(u_n)|}{(u_n + \frac{1}{n})^\theta} \psi \right\}_n$ is bounded in $L^1(\Omega_T)$ for all $\theta < 1$. Then, the claim is proved.

By similar computation as above, taking $\frac{\psi}{(u_n + \frac{1}{n})^\delta}$ as a test function in (40) with $\delta < 1$, we reach that

$$(44) \quad \delta \iint_{\Omega_T} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{\delta+1-m}} \psi + \iint_{\Omega_T} \frac{h_n}{(u_n + \frac{1}{n})^\delta} \psi \leq C \text{ for all } n.$$

We will prove the point (1) in the Theorem, that is $\nabla u_n \rightarrow \nabla u$ a.e in Ω_T where $u = v^{\frac{1}{m}}$.

To prove this assertion it is sufficient to show that, for some $s \in (0, 1)$,

$$(45) \quad \int_{\Omega} |\nabla T_k(u_n) - \nabla T_k(u)|^s \psi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Consider $A \equiv \{(x, t) \in \Omega_T : u(x, t) = 0\} \equiv \{(x, t) \in \Omega_T : v(x, t) = 0\}$. Notice that $\nabla u_n \rightarrow \nabla u$ in $\Omega_T \setminus A$. Since $\nabla T_k(u) = 0$ and $\nabla T_k(v) = 0$ in A , then

$$(46) \quad \iint_{\Omega_T} |\nabla T_k(u_n) - \nabla T_k(u)|^s \psi = \iint_{\{u=0\}} |\nabla T_k(u_n)|^s \psi + \iint_{\{u>0\}} |\nabla T_k(u_n) - \nabla T_k(u)|^s \psi$$

Since $s < 1$ and $\nabla u_n^m \rightarrow \nabla u^m$ strongly in the space $L^\sigma((0, T); L^\sigma(\Omega))$ for all $1 \leq \sigma < 1 + \frac{1}{Nm+1}$, it follows that

$$\iint_{\Omega_T \setminus A} |\nabla T_k(u_n) - \nabla T_k(u)|^s \psi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

To conclude the proof, it is sufficient to show that $\int_A |\nabla T_k(u_n)|^s \psi \rightarrow 0$ as $n \rightarrow \infty$.

Since $T_k(u_n) \rightarrow T_k(u)$ strongly in $L^\sigma(\Omega_T)$ for all $\sigma > 1$, then by Egorov's Lemma, for every $\epsilon > 0$, there exists a measurable set B_ϵ such that $|B_\epsilon| \leq \epsilon$ and $T_k(u_n) \rightarrow T_k(u)$ uniformly in $\Omega_T \setminus B_\epsilon$.

Then

$$\begin{aligned} \iint_{\{T_k(u)=0\}} |\nabla T_k(u_n)|^s \psi &= \iint_{\{T_k(u)=0\} \cap B_\varepsilon} |\nabla T_k(u_n)|^s + \iint_{\{T_k(u)=0\} \cap \Omega \setminus B_\varepsilon} |\nabla T_k(u_n)|^s \psi \\ &= I_1 + I_2 \end{aligned}$$

Using the fact that $\{|\nabla T_k(u_n)|\}_n$ is uniformly bounded in L^1 , and choosing $s < 1$, we find that

$$I_1 \leq C|B_\varepsilon|^{1-s}.$$

By using the uniform convergence of $T_k(u_n)$ in $\Omega_T \setminus B_\varepsilon$, we obtain

$$(47) \quad I_2 \leq \iint_{\{T_k(u_n) \leq M\} \cap \Omega \setminus B_\varepsilon} |\nabla T_k(u_n)|^s \psi \leq (M + \frac{1}{n})^{as} \iint_{\{T_k(u_n) \leq M\} \cap \Omega \setminus B_\varepsilon} \left(\frac{|\nabla T_k(u_n)|}{(u_n + \frac{1}{n})^a} \right)^s \psi$$

where $a > 0$. Since the estimate (42) holds, it is sufficient pick up $s < 1$ and $a > 0$ such that

$$\iint_{\{T_k(u_n) \leq M\} \cap \Omega \setminus B_\varepsilon} \left(\frac{|\nabla T_k(u_n)|}{(u_n + \frac{1}{n})^a} \right)^s \psi \leq C.$$

Taking limits for $M \rightarrow 0$ in (47) the result follows. \blacksquare

Remark 2.12. For the case where $0 < m \leq \frac{(N-2)_+}{N}$, the difficulty is to show the strong convergence of the sequence $\{u_n\}_n$ in $L^1(\Omega_T)$. This can be proved by assuming additional hypotheses on the structure of the measure μ , see Section 4. Once proving this strong convergence, the result of the Theorem (2.11) holds with the same conclusions.

3. THE POROUS MEDIUM EQUATION WITH GRADIENT TERM

The main goal of this section is to prove existence of solution to Problem (1). We start by obtaining *a priori estimates* for the truncated problems, in order to be able to apply the results for an associated *elliptic-parabolic* problem.

More precisely, the proof of existence follows the following steps.

- (1) We prove some a priori estimates that allow us to show that the right hand side of the truncated problems converge weak-* to a Radon measure.
- (2) We transform in a natural way the problem to an *elliptic-parabolic* problem.
- (3) By using the results of Theorem 2.11 and compactness arguments, we identify the measure limit as the second member of the Problem (1).

We divide the section in two parts according to the values of m .

3.1. The case $1 < m \leq 2$. More precisely, consider the problem

$$(48) \quad \begin{cases} u_t - \Delta u^m &= |\nabla u|^q + f(x, t) & \text{in } \Omega_T \equiv \Omega \times (0, T), \\ u(x, t) &\geq 0 & \text{in } \Omega_T, \\ u(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{if } x \in \Omega, \end{cases}$$

where $m > 1$, $q \leq 2$, $\Omega \subset \mathbb{R}^N$ a bounded domain, f and u_0 nonnegative functions under suitable hypotheses given below.

We will use as starting point the results in [12] for bounded data, $f \in L^\infty(\Omega_T)$ and $u_0 \in L^\infty(\Omega)$. Since $1 < m \leq 2$ and $1 \leq q \leq 2$ we will be able to obtain *a priori* estimates in the framework of [23] and [20], where *a priori* estimates are obtained to analyze the behavior of *viscosity supersolution*, to the porous medium equation. See [20] and [23] for the details concerning to this framework.

More precisely we have the next theorem.

Theorem 3.1. *Assume that $1 < m \leq 2$ and $q \leq 2$, then*

- (1) *If $q'(m-1) > 2$, $u_0 \in L^{1+\theta}(\Omega)$ and $f \in L^{1+\frac{2\theta}{mN}}(0, T; L^{\frac{(\theta+m)N}{mN+2\theta}}(\Omega))$ where $\theta \geq 2-m$. Then problem (48) has a distributional solution.*
- (2) *If $q'(m-1) \leq 2$*
 - (a) *If $q < m$, problem (48) has a solution for all f, u_0 as in the first case.*
 - (b) *If $m \leq q \leq 2$, then problem (48) has a solution if $e^{\alpha u_0} \in L^1(\Omega)$ for some $\alpha > 0$ and $f \in L^r(0, T; L^s(\Omega))$ where $1 < r < \infty, s > \frac{N}{2}$ and $\frac{1}{r} + \frac{N}{2s} = 1$.*

PROOF. STEP 1. A PRIORI ESTIMATES.

We prove separately each case.

(I). $q'(m-1) > 2$

Assume that $q'(m-1) > 2$ and fixed $\theta > 2-m$, since $m > 1$, then $q < 2$. Let $u_0 \in L^{1+\theta}(\Omega)$ and $f \in L^{1+\frac{2\theta}{mN}}(0, T; L^{\frac{(\theta+m)N}{mN+2\theta}}(\Omega))$, then there exist sequences, $\{f_n\}_n, \{u_{0n}\}_n$ such that $f_n \in L^\infty(\Omega_T), u_{0n} \in L^\infty(\Omega), u_{0n} \uparrow u_0$ in $L^{1+\theta}(\Omega)$ and $f_n \uparrow f$ in $L^{1+\frac{2\theta}{mN}}(0, T; L^{\frac{(\theta+m)N}{mN+2\theta}}(\Omega))$.

Define u_n , to be the bounded solution of the approximated problem

$$(49) \quad \begin{cases} u_{nt} - \operatorname{div}(m(u_n + \frac{1}{n})^{m-1} \nabla u_n) &= \frac{|\nabla u_n|^q}{|\nabla u_n|^q + \frac{1}{n}} + f_n & \text{in } \Omega_T, \\ u_n(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) &= u_{0n}(x) & \text{if } x \in \Omega. \end{cases}$$

Notice that the existence and the boundedness of u_n follow using the result of [12].

Taking $(G_k(u_n))^\theta$ as a test function in (49), with $\theta > 2-m$, it follows that

$$(50) \quad \begin{aligned} & \frac{d}{dt} \frac{1}{\theta+1} \int_{\Omega} (G_k(u_n))^{\theta+1} dx + m\theta \int_{\Omega} u_n^{m-1} (G_k(u_n))^{\theta-1} |\nabla u_n|^2 dx \leq \\ & \int_{\Omega} (G_k(u_n))^\theta |\nabla u_n|^q dx + \int_{\Omega} f_n (G_k(u_n))^\theta dx. \end{aligned}$$

Using the Hölder inequality,

$$(51) \quad \int_{\Omega} (G_k(u_n))^\theta |\nabla u_n|^q dx \leq \epsilon \int_{\Omega} (G_k(u_n))^{m+\theta-2} |\nabla u_n|^2 dx + c(\epsilon) \int_{\Omega} (G_k(u_n))^{\theta+\frac{q(2-m)}{2-q}} dx.$$

Since $q'(m-1) > 2$, then $\theta + \frac{q(2-m)}{2-q} < \theta + 1$. Thus

$$\int_{\Omega} (G_k(u_n))^\theta |\nabla u_n|^q dx \leq \epsilon \int_{\Omega} (G_k(u_n))^{m+\theta-2} |\nabla u_n|^2 dx + c(\epsilon) \int_{\Omega} (G_k(u_n))^{\theta+1} dx + C(\Omega).$$

We deal with the last term in (50). Using Hölder, Young and Sobolev inequalities we reach that

$$\begin{aligned} \int_{\Omega} f_n(G_k(u_n))^{\theta} dx &\leq \epsilon \left(\int_{\Omega} (G_k(u_n))^{\frac{(m+\theta)N}{N-2}} dx \right)^{\frac{N-2}{N}} + C(\epsilon) \left(\int_{\Omega} f^{\frac{(m+\theta)N}{mN+2\theta}} dx \right)^{\frac{mN+2\theta}{Nm}} \\ &\leq \frac{\epsilon}{S} \int_{\Omega} |\nabla(G_k(u))^{\frac{m+\theta}{2}}| dx + C(\epsilon) \left(\int_{\Omega} f^{\frac{(m+\theta)N}{mN+2\theta}} dx \right)^{\frac{mN+2\theta}{Nm}}. \end{aligned}$$

Choosing ϵ small enough, it follows that

$$\begin{aligned} (52) \quad &\frac{d}{dt} \frac{1}{\theta+1} \int_{\Omega} (G_k(u_n))^{\theta+1} dx + c \int_{\Omega} (G_k(u_n))^{\theta+m-2} |\nabla G_k(u_n)|^2 dx \leq \\ &c(\epsilon) \int_{\Omega} (G_k(u_n))^{\theta+1} dx + C(\epsilon) \left(\int_{\Omega} f^{\frac{(m+\theta)N}{mN+2\theta}} dx \right)^{\frac{mN+2\theta}{Nm}} + C(\Omega). \end{aligned}$$

Integrating in time and using Gronwall's lemma there results that

$$\begin{aligned} (53) \quad &C \frac{1}{\theta+1} \int_{\Omega} (G_k(u_n))^{\theta+1} dx + c \iint_{\Omega_T} (G_k(u_n))^{\theta+m-2} |\nabla G_k(u_n)|^2 dx \leq \\ &\frac{1}{\theta+1} \int_{\Omega} u_0^{\theta+1} dx + \int_0^T \left(\int_{\Omega} f^{\frac{(m+\theta)N}{mN+2\theta}} dx \right)^{\frac{mN+2\theta}{Nm}} + C(\Omega, T) \end{aligned}$$

Now, using $T_k(u_n)$ as a test function in the problem of u_n , we reach that

$$\begin{aligned} (54) \quad &\int_{\Omega} \Theta_k(u_n) dx + m \iint_{\Omega_T} (u_n + \frac{1}{n})^{m-1} |\nabla T_k(u_n)|^2 \\ &\leq \iint_{\Omega_T} T_k(u_n) |\nabla u_n|^q dx dt + k \iint_{\Omega_T} f \\ &\leq \iint_{\{u_n \leq \sigma\}} T_k(u_n) |\nabla u_n|^q dx dt + k \iint_{\{u_n \geq \sigma\}} |\nabla u_n|^q dx dt + k \iint_{\Omega_T} f \end{aligned}$$

where $\Theta_k(s) = \int_0^s T_k(\sigma) d\sigma$. From (53) we obtain that $\iint_{\{u_n \geq \sigma\}} |\nabla u_n|^q dx dt \leq C(\sigma, f)$, thus

$$\int_{\Omega} \Theta_k(u_n) dx + m \iint_{\Omega_T} (T_k(u_n))^{m-1} |\nabla T_k(u_n)|^2 \leq \iint_{\{u_n \leq \sigma\}} T_k(u_n) |\nabla u_n|^q dx dt + C(k, a, f).$$

Notice that, choosing $\sigma \ll k$ small, we get the existence of $C(k) \gg 1$ such that $T_k^{m-1}(s) \geq C(k)T_k(s)$, $0 \leq s \leq \sigma$, hence, using Young's inequality (if $q < 2$), there results that

$$(55) \quad \int_{\Omega} \Theta_k(u_n) dx + c \iint_{\Omega_T} T_k^{m-1}(u_n) |\nabla T_k(u_n)|^2 \leq C(\Omega, T, k).$$

Therefore, combining (53) and (55) we conclude that

- (1) $\{u_n^{1+\theta}\}_n$ is bounded in $L^\infty(0, T; L^1(\Omega))$.
- (2) $\{(G_k(u))_n^{\frac{\theta+m}{2}}\}_n$ is bounded in $L^2(0, T; W_0^{1,2}(\Omega))$ and then using Poincaré inequality, it follows that $\{u_n^{m+\theta}\}_n$ is bounded in $L^1(\Omega_T)$.

We claim that $\{|\nabla T_k(u_n)|\}_n$ is bounded in $L^2(\Omega_T)$ if $m < 2$, while, $\{|\nabla T_k(u_n)|\}_n$ is bounded in $L^2(\Omega_T, \delta(x))$ if $m = 2$ where $\delta(x) \equiv \text{dist}(x, \partial\Omega)$.

If $m < 2$, then using $w_n \equiv e^{\frac{c}{m(2-m)}(T_k(u_n) + \frac{1}{n})^{2-m}} - e^{\frac{c}{m(2-m)}(\frac{1}{n})^{2-m}}$ as a test function in (49), then we get

$$\begin{aligned} & \int_{\Omega} L_n(u_n) dx + c \iint_{\Omega_T} e^{\frac{c}{m(2-m)}(T_k(u_n) + \frac{1}{n})^{2-m}} |\nabla T_k(u_n)|^2 dx dt \leq \\ & \int_{\Omega} L_n(u_0) dx + \iint_{\Omega_T} e^{\frac{c}{m(2-m)}(T_k(u_n) + \frac{1}{n})^{2-m}} |\nabla u_n|^q dx dt + C(k) \iint_{\Omega_T} f dx \end{aligned}$$

where $L_n(s) = \int_0^s \left(e^{\frac{c}{m(2-m)}(T_k(s) + \frac{1}{n})^{2-m}} - e^{\frac{c}{m(2-m)}(\frac{1}{n})^{2-m}} \right) ds$. Notice that $L_n(s) \leq C(k)s$

Thus choosing $c \gg 1$ and using estimate (53) on $G_k(u_n)$ there results that

$$(56) \quad \iint_{\Omega_T} |\nabla T_k(u_n)|^2 dx dt \leq C(k).$$

Therefore the claim follows in this case.

Assume that $m = 2$ and consider ϱ , defined in (20). Fixed $0 < \alpha < 1$, to be chosen later, using $\frac{\varrho}{(u_n + \frac{1}{n})^\alpha}$ as a test function in (49), we get

$$\begin{aligned} & \frac{1}{1-\alpha} \int_{\Omega} (u_n + \frac{1}{n})^{1-\alpha} \varrho dx + \frac{1}{2-\alpha} \iint_{\Omega_T} \left((u_n + \frac{1}{n})^{2-\alpha} - (\frac{1}{n})^{2-\alpha} \right) = \\ & \alpha \iint_{\Omega_T} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^\alpha} \varrho + \iint_{\Omega_T} \frac{\varrho}{(u_n + \frac{1}{n})^\alpha} \frac{|\nabla u_n|^q}{|\nabla u_n|^q + \frac{1}{n}} + \\ & \iint_{\Omega_T} \frac{f_n \varrho}{(u_n + \frac{1}{n})^\alpha} + \frac{1}{1-\alpha} \int_{\Omega} (u_{0n} + \frac{1}{n})^{1-\alpha} \varrho dx. \end{aligned}$$

Choosing α such that $1-\theta < \alpha < 1$, then from (53), it follows that the first term in the above identity is uniformly bounded in n . Thus

$$\alpha \iint_{\Omega_T} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^\alpha} \varrho \leq C$$

and then

$$\iint_{\Omega_T} |\nabla T_k(u_n)|^2 \varrho \leq C(k).$$

Since $\phi \preceq \delta(x)$, the claim follows in this case.

Combining the result of the claim and estimates (53) and (55), we get easily that

- (1) $\{|\nabla u_n|^q\}_n$ is bounded in $L^1(\Omega_T)$ if $m < 2$,
- (2) $\{|\nabla u_n|^q\}_n$ is bounded in $L^1_{loc}(\Omega_T)$ if $m = 2$.

Notice that in both cases we can prove that $\{u_n^{\theta+m-2} |\nabla u_n|^2\}_n$ is bounded in $L^1(\Omega_T)$.

(II) $q'(m-1) \leq 2$ and $q < m$. We deal now with the case 2) - a). Assume that $q'(m-1) \leq 2$ and $q < m$, then the result follows using the same kind of computations as in the first case, the main

difficulty is to estimate the second term in (51) where we use Poincaré inequality. This is possible using the fact that $\theta + \frac{q(2-m)}{2-q} < m + \theta$.

Since $\theta + \frac{q(2-m)}{2-q} < m + \theta$, then using Hölder and Poincaré inequalities there result that

$$\begin{aligned} \int_{\Omega} (G_k(u_n))^{\theta + \frac{q(2-m)}{2-q}} dx &\leq \varepsilon \int_{\Omega} (G_k(u_n))^{m+\theta} dx + C(\varepsilon, \Omega) \\ &\leq \frac{\varepsilon}{\lambda_1} \int_{\Omega} |\nabla (G_k(u_n))^{\frac{m+\theta}{2}}|^2 dx + C(\varepsilon, \Omega). \end{aligned}$$

Choosing ε small enough and going back to estimate (53), it follows that

$$(57) \quad C \frac{1}{\theta+1} \int_{\Omega} (G_k(u_n))^{\theta+1} dx + c \iint_{\Omega_T} (G_k(u_n))^{\theta+m-2} |\nabla G_k(u_n)|^2 dx \leq C(u_0, f, T, \Omega).$$

(III) $q'(m-1) \leq 2$ and $m \leq q$.

Consider now the case where $q'(m-1) \leq 2$ and $m \leq q$. The existence result in this case follows using the same arguments as in [17], for the readers convenience we include here some details.

Using $e^{\alpha(G_k(u_n))} - 1$, with $\alpha > 0$, as a test function in the approximated problem for u_n , and calling $H_k(s) = \int_0^s (e^{\alpha(G_k(\sigma))} - 1) d\sigma$, we reach that

$$(58) \quad \begin{aligned} &\frac{d}{dt} \int_{\Omega} H_k(u_n) dx + m\alpha \int_{\Omega} u_n^{m-1} e^{\alpha(G_k(u_n))} |\nabla G_k(u_n)|^2 dx = \\ &\int_{\Omega} (e^{\alpha(G_k(u_n))} - 1) |\nabla u_n|^q dx + \int_{\Omega} f_n (e^{\alpha(G_k(u_n))} - 1) dx. \end{aligned}$$

Without loss of generality we can assume that $k \geq 1$.

a) If $q = 2$, choosing $m\alpha > 1$, it follows that

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} H_k(u_n) dx + m\alpha \int_{\Omega} (u_n^{m-1} - 1) e^{\alpha(G_k(u_n))} |\nabla u_n|^2 dx + \int_{\Omega} |\nabla G_k(u_n)|^2 dx \leq \\ &\int_{\Omega} f_n (e^{\alpha(G_k(u_n))} - 1) dx. \end{aligned}$$

b) If $q < 2$ by using Young inequality we obtain that

$$\int_{\Omega} (e^{\alpha(G_k(u_n))} - 1) |\nabla u_n|^q dx \leq \varepsilon \int_{\Omega} e^{\alpha(G_k(u_n))} |\nabla u_n|^2 dx + C(\varepsilon) \int_{\Omega} (e^{\alpha(G_k(u_n))} - 1) dx.$$

Hence

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} H_k(u_n) dx + m\alpha \int_{\Omega} (u_n^{m-1} - \varepsilon) e^{\alpha(G_k(u_n))} |\nabla u_n|^2 dx + \int_{\Omega} |\nabla G_k(u_n)|^2 dx \leq \\ &C(\varepsilon) \int_{\Omega} (e^{\alpha(G_k(u_n))} - 1) dx + \int_{\Omega} f_n (e^{\alpha(G_k(u_n))} - 1) dx. \end{aligned}$$

Fixed $k \geq 2$, then integrating in $[0, \tau]$ and taking the maximum on τ it follows that

$$(59) \quad \sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau)) dx + C \iint_{\Omega_T} u_n^{m-1} e^{\alpha(G_k(u_n))} |\nabla u_n|^2 \leq \int_{\Omega} H_k(u_{0n}(x)) dx + C \iint_{\Omega_T} (e^{\alpha(G_k(u_n))} - 1) + \iint_{\Omega_T} f_n(e^{\alpha(G_k(u_n))} - 1) \text{ if } q < 2.$$

As a consequence we find that

$$(60) \quad \iint_{\Omega_T} |\nabla G_k u_n|^2 \leq C.$$

Moreover,

$$(61) \quad \sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau)) dx + C \iint_{\Omega_T} u_n^{m-1} e^{\alpha(G_k(u_n))} |\nabla u_n|^2 \leq \int_{\Omega} H_k(u_{0n}(x)) dx + \iint_{\Omega_T} f_n(e^{\alpha(G_k(u_n))} - 1) \text{ if } q = 2.$$

Let us analyze the term $\iint_{\Omega_T} f_n(e^{\alpha(G_k(u_n))} - 1)$.

$$\begin{aligned} \iint_{\Omega_T} f_n(e^{\alpha(G_k(u_n))} - 1) &\leq \iint_{\Omega_T} f(e^{\frac{\alpha}{2}(G_k(u_n))} - 1)^2 + C \\ &\leq \|f\|_{r,s} \| (e^{\frac{\alpha}{2}(G_k(u_n))} - 1) \|_{r',s'}^2 + C. \end{aligned}$$

For simplicity of notation we set $w_n = e^{\frac{\alpha}{2}(G_k(u_n))} - 1$. Using the Gagliardo Nirenberg inequalities there results that

$$\|w_n\|_{r',s'}^2 \leq C \|w_n\|_{\infty,2}^{\frac{2}{r}} \left(\iint_{\Omega_T} |\nabla w_n|^2 \right)^{\frac{1}{r}'} \leq C \left(\sup_{\tau \in [0, T]} \int_{\Omega} w_n^2 dx \right)^{\frac{1}{r}'} \left(\iint_{\Omega_T} |\nabla w_n|^2 \right)^{\frac{1}{r}'}.$$

Hence using Young's inequality we reach

$$\iint_{\Omega_T} f_n(e^{\alpha(G_k(u_n))} - 1) dx \leq C(\varepsilon) \|f\|_{r,s}^{r'} \iint_{\Omega_T} |\nabla w_n|^2 + \varepsilon \left(\sup_{\tau \in [0, T]} \int_{\Omega} w_n^2 dx \right) + C.$$

Notice that $H_k(u_n) \geq c_1 w_n^2 - c_2$, then choosing ε small it follows that

$$\begin{aligned} \sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau)) dx + c(m\alpha - \varepsilon) \iint_{\Omega_T} u_n^{m-1} |\nabla w_n|^2 dx &\leq \\ C(\varepsilon) \|f\|_{r,s}^{r'} \iint_{\Omega_T} |\nabla w_n|^2 + C(T). \end{aligned}$$

If $\|f\|_{r,s}^{r'}$ is sufficiently small we get

$$\sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau)) dx + c(m\alpha - \varepsilon) \iint_{\Omega_T} u_n^{m-1} |\nabla w_n|^2 \leq C.$$

If not, then we can choose $t_1 < T$ such that $\|f\|_{r,s}'$ is sufficiently small, then

$$\sup_{\tau \in [0, t_1]} \int_{\Omega} H_k(u_n(x, \tau)) dx + c(m\alpha - \varepsilon) \int_0^{t_1} \int_{\Omega} u_n^{m-1} |\nabla w_n|^2 dx \leq C.$$

Then the general result follows by iteration. Hence we conclude that

$$\sup_{\tau \in [0, T]} \int_{\Omega} H_k(u_n(x, \tau)) dx + c \iint_{\Omega_T} u_n^{m-1} e^{\alpha(G_k(u_n))} |\nabla u_n|^2 \leq C(\Omega, T).$$

Now, taking $T_k(u_n)$ and using the previous estimate, it follows that

$$(62) \quad \int_{\Omega} \Theta_k(u_n) dx + c \iint_{\Omega_T} T_k^{m-1}(u_n) |\nabla T_k(u_n)|^2 \leq k \iint_{\Omega_T} f + C(\Omega, T, k).$$

Thus there results that $|\nabla u_n|^q + f_n$ is bounded in $L^1(\Omega_T)$.

STEP 2. PASSAGE TO THE LIMIT.

To obtain the existence of solution we need to prove that

$$(63) \quad \frac{|\nabla u_n|^q}{1 + \frac{1}{n} |\nabla u_n|^q} + f_n \rightarrow |\nabla u|^q + f \text{ strongly in } L^1_{loc}(\Omega_T).$$

Claim.- The following inequality holds,

$$(64) \quad \int_{\{u_n \leq M\}} \frac{|\nabla u_n|^q}{(1 + \frac{1}{n} |\nabla u_n|^q)(u_n + \frac{1}{n})^s} \varrho \leq C$$

for all $s < 1$, where ϱ is the solution of (20).

To prove the claim, consider $\frac{\varrho}{(u_n + \frac{1}{n})^s}$, with $s < 1$, as a test function in (49). Therefore

$$\begin{aligned} & \iint_{\Omega_T} (u_n)_t (u_n + \frac{1}{n})^{-s} \varrho + m \iint_{\Omega_T} (u_n + \frac{1}{n})^{m-1-s} \nabla u_n \nabla \varrho \geq \\ & s \iint_{\Omega_T} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{s+1}} \varrho + \iint_{\Omega_T} \frac{|\nabla u_n|^q}{(1 + \frac{1}{n} |\nabla u_n|^q)(u_n + \frac{1}{n})^s} \varrho \end{aligned}$$

and then

$$\iint_{\Omega_T} \frac{|\nabla u_n|^q}{(1 + \frac{1}{n} |\nabla u_n|^q)(u_n + \frac{1}{n})^s} \varrho \leq \int_{\Omega} \frac{1}{1-s} (u_n + \frac{1}{n})^{1-s} \varrho dx + \frac{1}{m-s} \iint_{\Omega_T} (u_n + \frac{1}{n})^{m-s}$$

Since $s < 1$ we obtain

$$(65) \quad \iint_{\Omega_T} \frac{|\nabla u_n|^q}{(1 + \frac{1}{n} |\nabla u_n|^q)(u_n + \frac{1}{n})^s} \varrho \leq C,$$

and the claim follows.

By using the *a priori* estimates of the first step, there results that

$$\frac{|\nabla u_n|^q}{1 + \frac{1}{n} |\nabla u_n|^q} + f_n \equiv h_n \text{ is bounded in } L^1(\Omega_T) \text{ (Resp. in } L^1_{loc}(\Omega_T)) \text{ if } m < 2 \text{ (Resp. if } m = 2).$$

Define now

$$v_n = (u_n + \frac{1}{n})^m - (\frac{1}{n})^m,$$

then v_n solves the problem,

$$(66) \quad \begin{cases} b(v_n)_t - \Delta v_n &= h_n, & (x, t) \in \Omega_T, \\ v_n(x, t) &= 0, & (x, t) \in \partial\Omega \times (0, T), \\ v_n(x, 0) &= \varphi^{-1}(T_n(u_0(x))), & x \in \Omega, \end{cases}$$

where $b(s) = (s + (\frac{1}{n})^m)^{\frac{1}{m}} - \frac{1}{n}$ satisfies the hypotheses (B).

Thus by Propositions 2.6 and 2.8, we have that

$$(67) \quad \nabla v_n \rightarrow \nabla v \text{ strongly in } L^\sigma(\Omega_T) \text{ for all } 1 \leq \sigma < 1 + \frac{1}{1 + Nm}.$$

To conclude we only need to prove (63).

The case $q = 2$ is treated in [17] by using some kind of exponential change of variables. We deal with the case $q < 2$.

Since $\{|\nabla T_k(u_n)|\}_n$ is bounded in $L^2(\Omega_T)$ and, by Theorem 2.11, $\nabla u_n \rightarrow \nabla u$ a.e. in Ω_T , then

$$\nabla T_k(u_n) \rightarrow \nabla T_k(u_n) \text{ strongly in } L^q(\Omega_T).$$

We will use Vitali's lemma. Consider $E \subset \Omega_T$, a measurable set, then we write,

$$\iint_E |\nabla u_n|^q dx dt = \iint_{E \cap \{u_n < k\}} |\nabla T_k(u_n)|^q dx dt + \iint_{E \cap \{u_n \geq k\}} |\nabla u_n|^q dx dt.$$

By using the strong convergence of the truncations, we have

$$\iint_E |\nabla T_k u_n|^q dx dt \rightarrow \iint_E |\nabla T_k u|^q dx dt.$$

We deal with the last term in the right hand side. By (53), (57) and (60) we have in all the cases,

$$\iint_{\{u_n \geq k\}} |\nabla u_n|^2 \leq C.$$

Then

$$\iint_{E \cap \{u_n \geq k\}} |\nabla u_n|^q dx dt \leq C \left(\iint_{\{u_n \geq k\}} |\nabla u_n|^2 \right)^{\frac{1}{2}} |\{u_n \geq k\}|^{\frac{1}{2}} \leq C |\{u_n \geq k\}|^{\frac{1}{2}}$$

It is clear that $|\{u_n \geq k\}| \rightarrow 0$ as $k \rightarrow \infty$ uniformly in n . Hence the result follows using Vitali's lemma.

If $m = 2$, then we can repeat the same arguments above to handle the term $|\nabla u_n|^q \psi$, for $\psi \in \mathcal{C}_0^\infty(\Omega_T)$.

Therefore in both cases we reach

$$\frac{|\nabla u_n|^q}{1 + \frac{1}{n} |\nabla u_n|^q} \rightarrow |\nabla u|^q \text{ strongly in } L_{loc}^1(\Omega_T)$$

and the existence result follows. ■

Remark 3.2. Notice that $1 + \theta$, $1 + \frac{2\theta}{mN}$, $\frac{(\theta+m)N}{mN+2\theta} \rightarrow 1$ as $\theta \rightarrow 0$. Then fixed $q < 2$, for all $\varepsilon > 0$, there exists $1 < m(\varepsilon) < 2$ such that if $m > m(\varepsilon)$, then problem (48) has a nonnegative solution for all $u_0 \in L^{1+\varepsilon}(\Omega)$ and $f \in L^{1+\varepsilon}(\Omega_T)$. This motivate the existence result studied in the next subsection.

3.2. The case $m > 2$: L^1 data.

In the elliptic case if $q(\frac{1}{m}-1) < -1$, then existence result holds for all L^1 data, without restriction on its size, see [2].

The goal of this subsection is to consider the case $m > 2$, that implies the above condition. In particular, we can also see the next result as a slight improvement of the result obtained in the elliptic case.

By using suitable a priori estimates, as in the elliptic case, we will prove that problem (48) has a distributional solution for all $f \in L^1(\Omega_T)$ and $u_0 \in L^1(\Omega)$. The main existence result is the following.

Theorem 3.3. *Let f, u_0 be such that $f \in L^1(\Omega_T)$ and $u_0 \in L^1(\Omega)$. Assume $1 < q \leq 2$ and $m > 2$, then problem (48) has a distributional solution u such that $|\nabla u^m| \in L_{loc}^\sigma(\Omega_T)$ for all $1 \leq \sigma < 1 + \frac{1}{Nm+1}$.*

PROOF. We will consider separately the cases $q < 2$ and $q = 2$.

Let begin by the case $q < 2$. Define u_n to be a solution to the approximated problem

$$(68) \quad \begin{cases} u_{nt} - \operatorname{div}(m(u_n + \frac{1}{n})^{m-1} \nabla u_n) &= \frac{|\nabla u_n|^q}{1 + \frac{1}{n} |\nabla u_n|^q} + T_n(f) & \text{in } \Omega_T, \\ u_n(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) &= T_n(u_0(x)) & \text{if } x \in \Omega. \end{cases}$$

Using $e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} \varrho$, where ϱ is defined in (20), as a test function in (68), it follows that

$$\begin{aligned} & \int_{\Omega} D_n(u_n) \varrho dx + c \iint_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^2 \varrho dx dt + \iint_{\Omega_T} K_n(u_n) dx dt \leq \\ & \int_{\Omega} D_n(u_{n0}) \varrho dx + \iint_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^q \varrho dx dt + \iint_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} \varrho T_n(f) dx dt \end{aligned}$$

with $D_n(s) = \int_0^s e^{-\frac{c}{(m-2)(t + \frac{1}{n})^{m-2}}} dt$ and $K_n(s) = \int_0^s m(s + \frac{1}{n})^{m-1} e^{-\frac{c}{(m-2)(t + \frac{1}{n})^{m-2}}} dt$

Since $e^{-\frac{c}{(m-2)s^{m-2}}} \leq C$, for $s \geq 0$, then $c_1 s - c_2 \leq D_n(s) \leq s$ and $K_n(s) \geq c_1 s^m - c_2$. Hence using Young's inequality,

$$\begin{aligned} & \int_{\Omega} D_n(u_n) \varrho dx + c \iint_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^2 \varrho dx dt + \iint_{\Omega_T} K_n(u_n) dx dt \leq \\ & \int_{\Omega} u_{n0} \varrho dx + \varepsilon \iint_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^2 \varrho dx dt + \\ & c \iint_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} \varrho dx dt + \iint_{\Omega_T} f dx dt. \end{aligned}$$

Choosing ε small it follows that

$$\int_{\Omega} D_n(u_n) \varrho dx + c \iint_{\Omega_T} e^{-\frac{c}{(m-2)(u_n + \frac{1}{n})^{m-2}}} |\nabla u_n|^2 \varrho dx dt + \iint_{\Omega_T} K_n(u_n) dx dt \leq C.$$

As a consequence, $\{u_n\}_n$ is bounded in $L^\infty(0, T; L_{loc}^1(\Omega))$, $\{u_n^m\}_n$ is bounded in $L^1(\Omega_T)$ and $\{G_k(u_n)\}_n$ is bounded in $L^2(0, T; W_{loc}^{1,2}(\Omega))$.

Using $T_k(u_n)\varrho$ as a test function in (68) we reach that the sequence $\{\Lambda_k(u_n)\}_n$ is bounded in the space $L^2(0, T; W_{loc}^{1,2}(\Omega))$ where $\Lambda_k(s) = \int_0^s (T_k(\sigma))^{\frac{m-1}{2}} d\sigma$.

In the same way we get easily that

$$(69) \quad \iint_{\Omega_T} \left(\frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} + T_n(f) \right) \varrho \leq C \text{ for all } n.$$

Fixed $s < \min\{1, m-2\}$, and using $\frac{\varrho}{(u_n + \frac{1}{n})^s}$ as a test function in (68) we reach that

$$(70) \quad \iint_{\Omega_T} (u_n + \frac{1}{n})^{m-2-\alpha} |\nabla u_n|^2 \varrho dxdt + \iint_{\Omega_T} \frac{T_n(f) \varrho}{(u_n + \frac{1}{n})^\alpha} dxdt \leq C \text{ for all } n$$

and

$$(71) \quad \iint_{\Omega_T} \left(\frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} \right) \frac{\varrho}{(u_n + \frac{1}{n})^s} \leq C \text{ for all } n.$$

Thus $\{u_n^{m-2-s} |\nabla u_n|^2\}_n$ is bounded in $L_{loc}^1(\Omega)$.

Notice that, if $2 < m < 3$, then there results that

$$\iint_{\Omega_T} |\nabla T_k(u_n)|^2 \varrho \leq C \text{ for all } n.$$

We claim that $G_k(u_n) \rightarrow G_k(u)$ strongly in $L^2(0, T; W_{loc}^{1,\alpha}(\Omega))$ for all $\alpha < 2$ and for all $k > 0$.

By Theorem 2.11, we have that $\nabla u_n \rightarrow \nabla u$ a.e in Ω_T in particular, $\nabla G_k(u_n) \rightarrow \nabla G_k(u)$ a.e in Ω_T . Hence to get the desired result we use Vitali's Lemma. Let $M > k$ and $\psi \in \mathcal{C}_0^\infty(\Omega_T)$, then for a measurable set $E \subset \Omega_T$ we have

$$\begin{aligned} & \iint_E |\nabla G_k(u_n)|^\alpha \psi = \\ & \iint_{E \cap \{u_n < M\}} |\nabla G_k(u_n)|^\alpha \psi + \iint_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^\alpha \psi = \\ & \iint_{E \cap \{k \leq u_n < M\}} |\nabla G_k(u_n)|^\alpha \psi + \iint_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^\alpha \psi. \end{aligned}$$

Since

$$(72) \quad T_k(u_n^m) \rightarrow T_k(u^m) \quad \text{strongly in } L^2(0, T; W_{loc}^{1,\alpha}(\Omega)) \text{ for all } \alpha < 2,$$

then

$$\iint_{E \cap \{k < u_n < M\}} |\nabla G_k(u_n)|^\alpha \psi \leq \frac{\varepsilon}{2} \text{ if } |E| \leq \delta_\varepsilon.$$

We deal now with the second term. Using (70) we have

$$\iint_{E \cap \{u_n \geq M\}} |\nabla u_n|^\alpha \psi dx \leq C \left(\iint_{\{u_n \geq M\}} u_n^{m-2-s} |\nabla u_n|^\alpha u_n^{-(m-2-s)} \psi dxdt \right) \leq \frac{C}{M^{m-2-s}}$$

where $s > 0$ is chosen such that $0 < s < m-2$. Thus choosing M large we reach that

$$\iint_{E \cap \{u_n \geq M\}} |\nabla u_n|^\alpha \psi dx \leq \frac{\varepsilon}{2}.$$

Therefore the strong convergence of $\{\nabla G_k(u_n)\}_n$ follows and the claim is proved.

As in the previous subsection, to get the existence result we just have to prove that

$$(73) \quad \frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} \rightarrow |\nabla u|^q \text{ strongly in } L^1_{loc}(\Omega_T).$$

From Theorem 2.11 and by the above estimate we reach $\frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} \psi \rightarrow |\nabla u|^q \psi$ a.e. in Ω_T .

Using (71), we can prove that

$$(74) \quad \limsup_{M \rightarrow 0} \iint_{\{u_n \leq M\}} \frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} \psi dx = 0 \text{ uniformly in } n.$$

Using the result of the last claim and from (72) we obtain that

$$\frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} \rightarrow |\nabla u|^q \text{ strongly in } L^1_{loc}(\Omega_T \cap \{u > 0\}).$$

If $|\{u = 0\}| = 0$, then (73) follows. Assume that $|\{u = 0\}| > 0$, since $|\nabla u|^q \in L^1_{loc}(\Omega_T)$, then we conclude that $|\nabla u|^q = 0$ on the set $\{u = 0\}$. Hence to finish the proof we have just to prove that

$$\frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} \rightarrow 0 \text{ strongly in } L^1_{loc}(\Omega_T \cap \{u = 0\}).$$

This follows using (74) and Egorov's theorem. Thus the existence result follows in this case.

We deal now with the case $q = 2$. From the result of [19], there exists a bounded solution u_n to the problem

$$(75) \quad \begin{cases} u_{nt} - \operatorname{div}(m(u_n + \frac{1}{n})^{m-1} \nabla u_n) &= |\nabla u_n|^2 + T_n(f) & \text{in } \Omega_T, \\ u_n(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) &= T_n(u_0(x)) & \text{if } x \in \Omega. \end{cases}$$

Using the same argument as in the case $q < 2$, we find the same estimates for the sequence $\{u_n\}_n$, moreover, for all $\psi \in \mathcal{C}^\infty_0(\Omega_T)$, $\psi \geq 0$,

$$(76) \quad \iint_{\Omega_T} |\nabla u_n|^2 \psi < C \text{ for all } n$$

and

$$(77) \quad \iint_{\Omega_T} \frac{|\nabla u_n|^2 \psi}{(u_n + \frac{1}{n})^s} \leq C \text{ for all } n.$$

By using Theorem 2.11, $\nabla u_n \rightarrow \nabla u$ e.a in Ω_T and then, as above,

$$(78) \quad T_k(u_n^m) \rightarrow T_k(u^m) \quad \text{strongly in } L^2(0, T; W^{1,\alpha}_{loc}(\Omega)) \text{ for all } \alpha < 2.$$

Now from (76), we conclude that

$$u_n \rightarrow u \quad \text{strongly in } L^2(0, T; W^{1,\alpha}_{loc}(\Omega)) \text{ for all } \alpha < 2.$$

Define $w_n \equiv T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - (T_k(u))_\nu$ where $(T_k(u))_\nu$ is defined as in (27) and $\gamma(s) = \frac{1}{m(2-m)}(s + \frac{1}{n})^{2-m}$. Let $h > 2k > 0$ to be chosen later. It is clear that $\nabla w_n \equiv 0$ for $u_n > M \equiv 4k + h$.

Using $w_n e^{\gamma(u_n)} \psi$ as a test function in (75), it follows that

$$\begin{aligned} & \int_0^T \langle (u_n)_t, e^{\gamma(u_n)} w_n \psi \rangle dt + m \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \psi \nabla T_M(u_n) \nabla w_n \\ & + m \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} w_n \nabla T_M(u_n) \nabla \psi \leq \iint_{\Omega_T} T_n(f) e^{\gamma(u_n)} w_n \psi \end{aligned}$$

It is clear that

$$\iint_{\Omega_T} T_n(f) e^{\gamma(u_n)} w_n \psi \leq \omega(n, \nu)$$

and

$$| \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} w_n \nabla T_M(u_n) \nabla \psi | \leq \omega(n, \nu).$$

It is well known that

$$\int_0^T \langle (u_n)_t, e^{\gamma(u_n)} w_n \psi \rangle dt \geq \omega(n) + \omega(\nu),$$

see, for instance, [17].

Let analyze the term $m \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla T_M(u_n) \nabla w_n \psi$.

$$\begin{aligned} & \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla T_M(u_n) \nabla w_n \psi \\ & = \iint_{\{u_n \leq k\}} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla T_k(u_n) \nabla w_n \psi + \iint_{\{u_n > k\}} e^{\gamma(u_n)} \psi (u_n + \frac{1}{n})^{m-1} \nabla T_M(u_n) \nabla w_n \\ & \geq \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla T_k(u_n) \nabla (T_k(u_n) - (T_k(u))_\nu)_+ \psi \\ & - \iint_{\{u_n > k\}} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} |\nabla T_M(u_n)| |\nabla (T_k(u))_\nu| \psi. \end{aligned}$$

Define

$$\Gamma_n(s) = \begin{cases} e^{\gamma(s)} (s + \frac{1}{n})^{m-1} & \text{if } s \leq k, \\ 0 & \text{if } s \geq k, \end{cases}$$

then

$$\begin{aligned} & \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla T_k(u_n) \nabla (T_k(u_n) - (T_k(u))_\nu)_+ \psi \\ & = \iint_{\Omega_T} \Gamma_n(u_n) \nabla T_k(u_n) \nabla (T_k(u_n) - (T_k(u))_\nu)_+ \psi \\ & = \iint_{\Omega_T} \Gamma_n(u_n) |\nabla (T_k(u_n) - (T_k(u))_\nu)_+|^2 \psi + \omega(n, \nu). \end{aligned}$$

On the other hand, for M fixed,

$$\iint_{\{u_n > k\}} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} |\nabla T_M(u_n)| |\nabla (T_k(u))_\nu| \psi = \omega(n, \nu).$$

Hence

$$\begin{aligned} \iint_{\Omega_T} \Gamma_n(u_n) |\nabla(T_k(u_n) - (T_k(u))_\nu)_+|^2 \psi &\leq \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla T_M(u_n) \nabla w_n \psi + \\ \iint_{\{u_n > k\}} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} |\nabla T_M(u_n)| |\nabla(T_k(u))_\nu| \psi &+ \omega(n, \nu). \end{aligned}$$

Thus

$$\iint_{\Omega_T} \Gamma_n(u_n) |\nabla(T_k(u_n) - (T_k(u))_\nu)_+|^2 \psi \leq \omega(n, \nu).$$

In the same way we reach that

$$\iint_{\Omega_T} \Gamma_n(u_n) |\nabla(T_k(u_n) - (T_k(u))_\nu)_-|^2 \psi \leq \omega(n, \nu).$$

Since $\Gamma_n(s) \geq C_1$ if $s > C_2 > 0$, uniformly in n , then

$$(79) \quad |\nabla u_n| \chi_{\{c_1 < u_n < c_2\}} \rightarrow |\nabla u| \chi_{\{c_1 < u < c_2\}} \text{ strongly in } L_{loc}^2(\Omega_T).$$

We claim that

$$(80) \quad |\nabla G_k(u_n)| \rightarrow |\nabla G_k(u)| \text{ strongly in } L_{loc}^2(\Omega_T).$$

We again use Vitali's lemma. Let $\psi \in \mathcal{C}_0^\infty(\Omega_T)$ be such that $\psi \geq 0$. Consider $E \subset \Omega_T$, a measurable set and write,

$$\begin{aligned} \iint_E |\nabla G_k(u_n)|^2 \psi &= \iint_{E \cap \{u_n \leq M\}} |\nabla G_k(u_n)|^2 \psi + \iint_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^2 \psi \\ &= \iint_{E \cap \{k \leq u_n \leq M\}} |\nabla G_k(u_n)|^2 \psi + \iint_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^2 \psi. \end{aligned}$$

From (79), given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n \rightarrow \infty} \iint_{E \cap \{k \leq u_n \leq M\}} |\nabla G_k(u_n)|^2 \psi \leq \frac{\epsilon}{2} \text{ if } |E| \leq \delta.$$

We deal now with the term $\int_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^2 \psi$.

Using (70), we get, for some $\alpha < m - 2$,

$$\iint_{E \cap \{u_n \geq M\}} |\nabla G_k(u_n)|^2 \psi = \iint_{E \cap \{u_n \geq M\}} \frac{(u_n + \frac{1}{n})^{m-2-\alpha}}{(u_n + \frac{1}{n})^{m-2-\alpha}} |\nabla u_n|^2 \psi \leq \frac{C}{(M + \frac{1}{n})^{m-2-\alpha}}.$$

Thus

$$\limsup_{n \rightarrow \infty} \iint_E |\nabla G_k(u_n)|^2 \psi \leq \frac{\epsilon}{2} + \frac{C}{(M + \frac{1}{n})^{m-2-\alpha}}.$$

Letting $M \rightarrow \infty$, we reach the claim.

In the same way and by using estimates (77), we can prove that

$$\iint_{\Omega_T} |\nabla(T_k(u_n) - T_k(u))|^2 \psi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus

$$|\nabla u_n|^2 \rightarrow |\nabla u|^2 \text{ strongly in } L_{loc}^1(\Omega_T),$$

and then the existence result follows. ■

Remark 3.4. If $m = q = 2$ and $f = 0$, we can prove that the solution to problem (48) has the finite speed propagation property. This follows by setting $w = \frac{2}{3} \left(\frac{4}{5} \right)^{\frac{2}{5}} u^{\frac{5}{2}}$, then w solves

$$(81) \quad \begin{cases} w_t - \frac{4}{5} \left(\frac{3}{2} \right)^{\frac{5}{3}} \Delta w^{\frac{5}{3}} &= 0 & \text{in } \Omega_T, \\ w(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ w(x, 0) &= \frac{2}{3} \left(\frac{4}{5} \right)^{\frac{2}{5}} u_0^{\frac{5}{2}}(x) & \text{in } \Omega. \end{cases}$$

If $u_0 \in L^\infty(\Omega)$ has a compact support, by using a convenient Barenblatt self-similar supersolution (see [25], for instance) we obtain the finite speed of propagation property. The inverse change of variable allow us to conclude the same result for problem

$$(82) \quad \begin{cases} u_t - \Delta u^2 &= |\nabla u|^2 & \text{in } \Omega_T, \\ u(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega. \end{cases}$$

4. THE FAST DIFFUSION EQUATION

In this section we consider the case $0 < m < 1$, usually called *fast diffusion equation* in the literature. We will prove the following existence result.

Theorem 4.1. Assume that $0 < m < 1$, $q \leq 2$ and

- (1) $f \in L^r(0, T; L^s(\Omega))$ where $1 < r < \infty$, $s > \frac{N}{2}$ with $\frac{1}{r} + \frac{N}{2s} = 1$
- (2) $e^{\alpha u_0^{2-m}} \in L^1(\Omega)$ where,
 - (a) either $\alpha > 0$ is any positive constant if $q < 2$
 - (b) or $\alpha m(2 - m) > 2$ if $q = 2$.

Then problem (48) has a distributional solution

PROOF. Let $\{f_n\}_n, \{u_{0n}\}_n$ be sequences of bounded nonnegative functions such that $u_{0n} \uparrow u_0$ and $f_n \uparrow f$.

Let u_n be the bounded solution of

$$(83) \quad \begin{cases} u_{nt} - \operatorname{div}(m(u_n + \frac{1}{n})^{m-1} \nabla u_n) &= \frac{|\nabla u_n|^q}{1 + \frac{1}{n} |\nabla u_n|^q} + f_n & \text{in } \Omega_T, \\ u_n(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) &= u_{0n}(x) & \text{if } x \in \Omega, \end{cases}$$

with data (f_n, u_{0n}) . Notice that the existence and the boundedness of u_n follow using the result of [12].

Taking $e^{\alpha u_n^{2-m}} - 1$, $\alpha > 0$, as a test function in (83), we find that

$$(84) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} H(u_n) dx + m\alpha(2 - m) \int_{\Omega} e^{\alpha u_n^{2-m}} |\nabla u_n|^2 dx \leq \\ & \int_{\Omega} (e^{\alpha u_n^{2-m}} - 1) |\nabla u_n|^q dx + \int_{\Omega} f_n (e^{\alpha u_n^{2-m}} - 1) dx \end{aligned}$$

where $H(s) = \int_0^s (e^{\alpha\sigma^{2-m}} - 1)d\sigma$.

Using Young inequality, integrating in $[0, \tau]$ and taking the maximum on τ

$$\begin{aligned} & \sup_{\tau \in [0, T]} \int_{\Omega} H(u_n(x, \tau))dx + (\alpha m(2-m) - \varepsilon) \iint_{\Omega_T} e^{\alpha u_n^{2-m}} |\nabla u_n|^2 \leq \\ & C \iint_{\Omega_T} (e^{\alpha u_n^{2-m}} - 1) + \iint_{\Omega_T} f_n(e^{\alpha u_n^{2-m}} - 1)dx + \int_{\Omega} H(u_0(x))dx \text{ if } q < 2 \end{aligned}$$

and

$$\begin{aligned} & \sup_{\tau \in [0, T]} \int_{\Omega} H(u_n(x, \tau))dx + (\alpha m(2-m) - 1) \iint_{\Omega_T} e^{\alpha u_n^{2-m}} |\nabla u_n|^2 dx \leq \\ & C \iint_{\Omega_T} (e^{\alpha u_n^{2-m}} - 1) + \int_{\Omega} f_n(e^{\alpha u_n^{2-m}} - 1)dx + \int_{\Omega} H(u_0(x))dx \text{ if } q = 2. \end{aligned}$$

Let us analyze the last term in (84).

$$\begin{aligned} \iint_{\Omega_T} f_n(e^{\alpha u_n^{2-m}} - 1)dx & \leq \iint_{\Omega_T} f(e^{\frac{\alpha}{2} u_n^{2-m}} - 1)^2 + C \\ & \leq \|f\|_{r,s} \| (e^{\frac{\alpha}{2} u_n^{2-m}} - 1) \|_{r',s'}^2 + C \end{aligned}$$

We set $w_n = e^{\frac{\alpha}{2} u_n^{2-m}} - 1$, then using the Gagliardo-Nirenberg inequality we obtain that

$$\|w_n\|_{r',s'}^2 \leq C \|w_n\|_{\infty,2}^{\frac{2}{r}} \left(\int_0^\tau \int_{\Omega} |\nabla w_n|^2 \right)^{\frac{1}{r}} \leq C \left(\sup_{\tau \in [0, T]} \int_{\Omega} w_n^2 dx \right)^{\frac{1}{r}} \left(\int_0^\tau \int_{\Omega} |\nabla w_n|^2 \right)^{\frac{1}{r}}.$$

Thus

$$\iint_{\Omega_T} f_n(e^{\alpha u_n^{2-m}} - 1)dx \leq C(\varepsilon) \|f\|_{r,s}^{r'} \int_0^\tau \int_{\Omega} |\nabla w_n|^2 + \varepsilon \left(\sup_{\tau \in [0, T]} \int_{\Omega} w_n^2 dx \right) + C.$$

Using the fact that $H(u_n) \geq c_1 w_n^2 - c_2$, then choosing ε small it follows that

$$\begin{aligned} & \sup_{\tau \in [0, T]} \int_{\Omega} H(u_n(x, \tau))dx + c(\alpha m(2-m) - \varepsilon) \iint_{\Omega_T} |\nabla w_n|^2 dx \leq \\ & C(\varepsilon) \|f\|_{r,s}^{r'} \int_0^\tau \int_{\Omega} |\nabla w_n|^2 + C(\Omega, T). \end{aligned}$$

If $\|f\|_{r,s}^{r'}$ is sufficiently small we get

$$\sup_{\tau \in [0, T]} \int_{\Omega} H(u_n(x, \tau))dx + c(\alpha m(2-m)\varepsilon) \iint_{\Omega_T} |\nabla w_n|^2 dx \leq C.$$

If not, then we can choose $t_1 < T$ such that $\|f\|_{r,s}^{r'}$ is sufficiently small, then

$$\sup_{\tau \in [0, t_1]} \int_{\Omega} H(u_n(x, \tau))dx + c(\alpha m(2-m) - \varepsilon) \int_0^{t_1} \int_{\Omega} |\nabla w_n|^2 dx \leq C.$$

Then the general result follows by iteration. Hence we conclude that

$$(85) \quad \sup_{\tau \in [0, T]} \int_{\Omega} H(u_n(x, \tau)) dx + c \iint_{\Omega_T} e^{\alpha u_n^{2-m}} |\nabla u_n|^2 dx \leq C(\Omega, T).$$

Therefore we conclude that $|\nabla u_n|^q + f_n$ is bounded in $L^1(\Omega_T)$.

It is clear from the above estimate that $\{u_n\}_n$ is bounded in the spaces $L^2(0, T; W_0^{1,2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ and $\{u_{nt}\}_n$ is bounded in the spaces $L^2(0, T; W^{-1,2}(\Omega)) + L^1(\Omega_T)$. Then, there exists a measurable function $u \in L^2(0, T; W_0^{1,2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ such that, up to a subsequence, $u_n \rightarrow u$ strongly in $C(0, T; L^2(\Omega))$. Hence using by the result of Theorem 2.11 and Remark 2.12, we get $\nabla u_n \rightarrow \nabla u$ a.e in Ω_T . Thus if $q < 2$, then by the previous estimate we obtain that $|\nabla u_n|^q \rightarrow |\nabla u|^q$ strongly in $L^1(\Omega_T)$ and then the result follows.

We deal now with the case $q = 2$. Consider $\psi \in C_0^\infty(\Omega_T)$, $\psi \geq 0$. By using $\frac{\psi}{(u_n + \frac{1}{n})^\delta}$ as a test function in (83), where $\delta < m$, there results that

$$(86) \quad \iint_{\Omega_T} \frac{|\nabla u_n|^2}{(u_n + \frac{1}{n})^{2+\delta-m}} \psi \leq C \text{ for all } n \text{ and for all } n.$$

Notice that from (86), it follows that

$$(87) \quad \iint_{\{u_n \leq M\}} |\nabla u_n|^2 \psi \leq C(M + \frac{1}{n})^{2+\delta-m} \text{ uniformly in } n,$$

and

$$(88) \quad \iint_{\Omega_T} \frac{|\nabla T_k(u_n)|^2}{(u_n + \frac{1}{n})^{2+\delta-m}} \psi \leq C \text{ for all } n.$$

Let us prove now that

$$(89) \quad |\nabla T_k(u_n)| \rightarrow |\nabla T_k(u)| \text{ strongly in } L_{loc}^2(\Omega_T).$$

Let φ be a real differentiable function such that $\varphi(0) = 0$ and $(2k)^{m-1}\varphi' - |\varphi| \geq C > 0$. Consider $w_n \equiv e^{-\gamma(T_k(u_n))} \varphi(T_n(u_n) - (T_k(u))_\nu)_+$ where $(T_k(u))_\nu$ is defined as in (27) and

$$\gamma(s) = \frac{1}{m(2-m)} \left((s + \frac{1}{n})^{2-m} - (\frac{1}{n})^{2-m} \right).$$

Using $w_n e^{\gamma(u_n)} \psi$ as a test function in (83), it follows that

$$\begin{aligned} & \int_0^T \langle (u_n)_t, e^{\gamma(u_n)} w_n \psi \rangle dt + m \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} \nabla u_n \nabla w_n \psi \\ & + m \iint_{\Omega_T} e^{\gamma(u_n)} (u_n + \frac{1}{n})^{m-1} w_n \nabla u_n \nabla \psi \leq \iint_{\Omega_T} f e^{\gamma(u_n)} w_n \psi \end{aligned}$$

Notice that

$$\int_0^T \langle (u_n)_t, e^{\gamma(u_n)} w_n \psi \rangle dt \geq \omega(n) + \omega(\nu).$$

(see for instance [17]). Using the hypothesis on f and by (85) it follows that

$$\iint_{\Omega_T} f e^{\gamma(u_n)} w_n \psi \leq \omega(n, \nu).$$

On the other hand,

$$\begin{aligned} & \left| \iint_{\Omega_T} e^{\gamma(u_n)} \left(u_n + \frac{1}{n}\right)^{m-1} w_n \nabla u_n \nabla \psi \right| \leq \\ & \left(\iint_{\text{Supp}(\psi)} e^{\gamma(u_n)} \left(u_n + \frac{1}{n}\right)^{2(m-1)} |\nabla u_n|^2 \right)^{\frac{1}{2}} \left(\iint_{\text{Supp}(\psi)} e^{\gamma(u_n)} w_n^2 |\nabla \psi|^2 \right)^{\frac{1}{2}} \leq \\ & C \left(\iint_{\text{Supp}(\psi)} e^{\gamma(u_n)} w_n^2 |\nabla \psi|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Notice that

$$e^{\gamma(u_n)} w_n^2 |\nabla \psi|^2 \rightarrow 0 \quad \text{a.e. in } \Omega_T \quad \text{and} \quad e^{\gamma(u_n)} w_n^2 |\nabla \psi|^2 \leq C e^{\gamma(u_n)},$$

therefore by the Sobolev inequality and the estimates (85) and (86), we reach that

$$\left(\iint_{\text{Supp}(\psi)} e^{\gamma(u_n)} w_n^2 |\nabla \psi|^2 \right)^{\frac{1}{2}} = \omega(n).$$

Therefore we conclude that

$$m \iint_{\Omega_T} e^{\gamma(u_n)} \left(u_n + \frac{1}{n}\right)^{m-1} \nabla u_n \nabla w_n \psi \leq \omega(n) + \omega(\nu).$$

Notice that,

$$\begin{aligned} & m \iint_{\Omega_T} e^{\gamma(u_n)} \left(u_n + \frac{1}{n}\right)^{m-1} \nabla u_n \nabla w_n \psi = \\ & m \iint_{\Omega_T} e^{\gamma((u_n)) - \gamma(T_k(u_n))} \left(u_n + \frac{1}{n}\right)^{m-1} \nabla u_n \nabla (T_k(u_n) - T_k(u))_+ \varphi'((T_k(u_n) - T_k(u))_+) \psi \\ & - m \iint_{\{u_n \leq k\}} |\nabla u_n|^2 \varphi((T_k(u_n) - T_k(u))_+) \psi \end{aligned}$$

Let us analyze each term in the previous identity.

$$\begin{aligned} & m \iint_{\Omega_T} e^{\gamma((u_n)) - \gamma(T_k(u_n))} \left(u_n + \frac{1}{n}\right)^{m-1} \nabla u_n \nabla (T_k(u_n) - T_k(u))_+ \varphi'((T_k(u_n) - T_k(u))_+) \psi = \\ & m \iint_{\{u_n \leq k\}} \left(u_n + \frac{1}{n}\right)^{m-1} |\nabla (T_k(u_n) - (T_k(u))_\nu)_+|^2 \varphi'((T_k(u_n) - T_k(u))_+) \psi \\ & + m \iint_{\{u_n \leq k\}} \left(u_n + \frac{1}{n}\right)^{m-1} \nabla((T_k(u))_\nu) \nabla (T_k(u_n) - (T_k(u))_\nu)_+ \varphi'((T_k(u_n) - T_k(u))_+) \psi \\ & - m \iint_{\{u_n \geq k\}} \left(u_n + \frac{1}{n}\right)^{m-1} \nabla((T_k(u))_\nu) \nabla (T_k(u_n) - (T_k(u))_\nu)_+ \varphi'((T_k(u_n) - T_k(u))_+) \psi \end{aligned}$$

It is clear that

$$|\iint_{\{u_n \geq k\}} (u_n + \frac{1}{n})^{m-1} \nabla((T_k(u))_\nu) \nabla(T_k(u_n) - (T_k(u))_\nu)_+ \varphi'((T_k(u_n) - T_k(u))_+) \psi| \leq \omega(n, \nu).$$

Now,

$$\begin{aligned} & \iint_{\{u_n \leq k\}} (u_n + \frac{1}{n})^{m-1} \nabla((T_k(u))_\nu) \nabla(T_k(u_n) - (T_k(u))_\nu)_+ \varphi'((T_k(u_n) - T_k(u))_+) \psi = \\ & \iint_{\{u_n \leq k\} \cap \{T_k(u_n) \geq (T_k(u))_\nu\}} \psi (u_n + \frac{1}{n})^{m-1} \nabla((T_k(u))_\nu) \nabla(T_k(u_n) - (T_k(u))_\nu)_+ \varphi'((T_k(u_n) - T_k(u))_+) \end{aligned}$$

Since $m < 1$, by using the dominated convergence theorem and by estimate (88),

$$\psi (u_n + \frac{1}{n})^{m-1} |\nabla((T_k(u))_\nu)| \chi_{\{T_k(u_n) \geq (T_k(u))_\nu\}} \rightarrow u^{m-1} |\nabla T_k(u)| \psi \text{ strongly in } L^2(\Omega_T) \text{ as } n, \nu \rightarrow \infty.$$

Thus using a duality argument, we reach

$$m \iint_{\{u_n \leq k\}} (u_n + \frac{1}{n})^{m-1} \nabla((T_k(u))_\nu) \nabla(T_k(u_n) - (T_k(u))_\nu)_+ \varphi'((T_k(u_n) - T_k(u))_+) \psi = \omega(n, \nu).$$

It is clear that

$$\begin{aligned} & -m \iint_{\{u_n \leq k\}} |\nabla u_n|^2 \varphi((T_k(u_n) - T_k(u))_+) \psi \geq \\ & -m \iint_{\{u_n \leq k\}} |\nabla(T_k(u_n) - (T_k(u))_\nu)_+|^2 \varphi((T_k(u_n) - T_k(u))_+) \psi + \omega(n) + \omega(\nu). \end{aligned}$$

Thus combining the above estimates, there results that

$$m \iint_{\{u_n \leq k\}} \left((u_n + \frac{1}{n})^{m-1} \varphi' - \varphi \right) |\nabla(T_k(u_n) - ((T_k(u))_\nu)_+)|^2 \leq \omega(n) + \omega(\nu)$$

and by the properties of φ ,

$$\iint_{\Omega_T} |\nabla(T_k(u_n) - T_k(u))_+|^2 \psi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In a similar way, we obtain

$$\iint_{\Omega_T} |\nabla(T_k(u_n) - T_k(u))_-|^2 \psi \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence

$$|\nabla T_k(u_n)| \rightarrow |\nabla T_k(u)| \text{ strongly in } L^2_{loc}(\Omega_T).$$

Thus using estimate (86), (89) and Vitali's lemma, we can prove that

$$|\nabla u_n| \rightarrow |\nabla u| \text{ strongly in } L^2_{loc}(\Omega_T).$$

and then the existence result follows. ■

4.1. Finite time extinction. Assume that $f \equiv 0$ and $q = 2$, that is, we consider the problem

$$(90) \quad \begin{cases} u_t - \Delta u^m &= |\nabla u|^2 & \text{in } \Omega_T, \\ u(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{if } x \in \Omega, \end{cases}$$

where $0 < m < 1$.

We will prove that the *regular* solutions of (90) become zero in finite time, provided the initial datum $u_0 \in L^{1+\theta_0}(\Omega)$ for some $\theta_0 > 0$.

We begin by defining the meaning of *regular* solutions to (90).

Definition 4.2. Let $H(s) = \int_0^s e^{\frac{2-m}{m(2-m)}t} dt$ and define

$$(91) \quad \beta(s) = \frac{1}{m} \int_0^s (H^{-1}(\sigma))^{\frac{1}{m}-1} d\sigma,$$

we say that u is a regular solution to problem (90) in Ω_T if

$$v \equiv H(u^m) \in L^2((0, T); W_0^{1,2}(\Omega)) \cap \mathcal{C}([0, T]; L^2(\Omega)), \quad \beta(v)_t \in L^2((0, T); W^{-1,2}(\Omega))$$

and for all $\phi \in L^2((0, T); W_0^{1,2}(\Omega))$ we have

$$(92) \quad \int_0^T \langle (\beta(v))_t, \phi \rangle + \int_0^T \int_{\Omega} \nabla v \cdot \nabla \phi = 0.$$

It is clear that the existence of a regular solution follows using Theorem 4.1 for $q = 2$, $f = 0$ and the regularity of u_0 .

We are able now to state the next result.

Theorem 4.3. Assume that $0 < m < 1$. If u is the regular solution of problem (90) in the sense of Definition 4.2, then there exists a positive, finite time t_0 , depending on N , and u_0 such that $u(x, t) \equiv 0$ for $t > t_0$.

PROOF. To get the desired result we have just to show that $v(x, t) \equiv 0$ for $t > t_0$. It is clear that v solves

$$(93) \quad \begin{cases} (\beta(v))_t - \Delta v &= 0 & \text{in } \Omega \times (0, T), \\ v(x, t) &= 0 & \text{on } \partial\Omega \times (0, T), \\ v(x, 0) &= v_0(x) & \text{in } \Omega, \end{cases}$$

with $v_0 \in L^2(\Omega)$. Using v^θ , where $\theta > 0$ to be chosen later, as a test function in (93), there result that

$$(94) \quad \frac{d}{dt} \int_{\Omega} \Psi(v(x, t)) dx + c(\theta) \int_{\Omega} |\nabla v^{\frac{\theta+1}{2}}| dx = 0,$$

where

$$\Psi(s) = \int_0^s s^\theta (H^{-1}(\sigma))^{\frac{1}{m}-1} d\sigma.$$

Since

$$\lim_{s \rightarrow \infty} \frac{H^{-1}(s)}{s^\varepsilon} = 0 \text{ for all } \varepsilon > 0$$

it follows that

$$\Psi(s) \leq c(\varepsilon)s^{\theta+1+\varepsilon(\frac{1}{m}-1)} \quad \text{for every } s \geq 0.$$

Fixed θ such that $\theta + 1 + \varepsilon(\frac{1}{m} - 1) = 1 + \theta_0$, then using Sobolev's and Hölder's inequalities,

$$\int_{\Omega} |\nabla v^{\frac{\theta+1}{2}}|^2 dx \geq c_1(N, \theta) \left[\int_{\Omega} (v^{\frac{\theta+1}{2}})^{2^*} dx \right]^{2/2^*} \geq c_2(N, \theta, |\Omega|) \left[\int_{\Omega} (v^{a(\theta+1)}) dx \right]^{1/a}$$

where $1 < a < \frac{2^*}{2}$ is chosen such that $a(\theta + 1) = \theta + 1 + \varepsilon(\frac{1}{m} - 1)$. Hence it follows that

$$\int_{\Omega} |\nabla v^{\frac{\theta+1}{2}}|^2 dx \geq c(N, \theta, |\Omega|) \left[\int_{\Omega} \Psi(v(x, t)) dx \right]^{1/a}.$$

Define

$$\xi(t) = \int_{\Omega} \Psi(v(x, t)) dx,$$

then

$$\frac{\xi'(t)}{\xi(t)^{1/a}} \leq -c_4 < 0.$$

Note that by the assumption on v_0 we reach that $\xi(0) < \infty$. Integrating in t , one obtains

$$\frac{a}{a-1} (\xi(t)^{\frac{a-1}{a}} - \xi(0)^{\frac{a-1}{a}}) \leq -c_4 t.$$

Thus, as long as $\xi(t) > 0$, one has

$$\xi(t)^{\frac{a-1}{a}} \leq \xi(0)^{\frac{a-1}{a}} - c_4 \frac{a-1}{a} t.$$

Therefore, $\xi(t) \equiv 0$ for t large enough. ■

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